uncertainties. However, the cost of carrying intervals (instead of simple machine numbers) throughout a lengthy computation may make the procedure cumbersome. Consequently, it is used only when great reliance must be placed on the computations. Also, it may be difficult to keep the intervals from growing much larger than is realistic. Recently, a number of software packages have been developed that have contributed significantly to the use of interval arithmetic in computations. Interval arithmetic has its own literature, including a journal devoted to research. Some books concerning it are Alefeld and Grigorieff [1980], Alefeld and Herzberger [1983], Kulisch and Miranker [1981], and Moore [1966, 1979]. Recent research developments on interval arithmetic can be found in a *homepage* on the Internet. (See Appendix A, "An Overview of Mathematical Software," p. 768.)

PROBLEMS 2.2

1. By using the error term in Taylor's Theorem, show that at least seven terms are required in the series of Example 2 if the error is not to exceed 10^{-9}.
2. How many bits of precision are lost in a computer when we carry out the subtraction $x - \sin x$ for $x = \frac{1}{2}$?
3. How many bits of precision are lost in the subtraction $1 - \cos x$ when $x = \frac{1}{4}$?
4. (Continuation) For the function in the preceding problem, find a suitable Taylor series by which it can be accurately computed.
5. Find a suitable trigonometric identity so that $1 - \cos x$ can be accurately computed for small x with calls to the system functions for $\sin x$ or $\cos x$. (There are two good answers.)
6. Find a way of computing $\sqrt{x^4 + 4} - 2$ without undue loss of significance.
7. Using the definition $\sinh x \equiv \frac{1}{2}(e^x - e^{-x})$, discuss the problem of computing $\sinh x$.
8. In solving the quadratic equation $ax^2 + bx + c = 0$ by use of the formula

$$x = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$$

there is a loss of significance when $4ac$ is small relative to b^2 because then

$$\sqrt{b^2 - 4ac} \approx |b|$$

Suggest a method to circumvent this difficulty.

9. Suggest ways to avoid loss of significance in these calculations.
   - **a.** $\sqrt{x^2 + 1} - x$
   - **b.** $\log x - \log y$
   - **c.** $x^{-3}(\sin x - x)$
   - **d.** $\sqrt{x + 2} - \sqrt{x}$
   - **e.** $e^x - e$
   - **f.** $\log x - 1$
   - **g.** $(\cos x - e^{-x})/\sin x$
   - **h.** $\sin x - \tan x$
   - **i.** $\sinh x - \tanh x$
   - **j.** $\ln\left(x + \sqrt{x^2 + 1}\right)$ *Hint:* This is the function $\sinh^{-1} x$.

10. For any $x_0 > -1$, the sequence defined recursively by

$$x_{n+1} = 2^{n+1}\left[\sqrt{1 + 2^{-n}x_n} - 1\right]$$

converges to $\ln(x_0 + 1)$. (See Henrici [1962, p. 243].) Arrange this formula in a way that avoids loss of significance.

11. Arrange the following formulas in order of merit for computing $\tan x - \sin x$ when x is near 0.

a. $\sin x[(1/\cos x) - 1]$

b. $\frac{1}{2}x^3$

c. $(\sin x)/(\cos x) - \sin x$

d. $(x^2/2)(1 - x^2/12)\tan x$

e. $\frac{1}{2}x^2 \tan x$

f. $\tan x \sin^2 x/(\cos x + 1)$

12. Find ways to compute these functions without serious loss of significance.

a. $(1 - x)/(1 + x) - 1/(3x + 1)$

b. $\sqrt{x + (1/x)} - \sqrt{x - (1/x)}$

c. $e^x - \cos x - \sin x$

13. Discuss the calculation of e^{-x} for $x > 0$ from the series

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

Suggest a better way, assuming that the system function for e^x is *not* available.

14. There is subtractive cancellation in computing $f(x) = 1 + \cos x$ for some values of x. What are these values of x and how can the loss of precision be averted?

15. Consider the function $f(x) = x^{-1}(1 - \cos x)$.

a. What is the *correct* definition of $f(0)$; that is, the value that makes f continuous?

b. Near what points is there a loss of significance if the given formula is used?

c. How can we circumvent the difficulty in part **b**? Find a method that does not use the Taylor series.

d. If the new formula that you gave in part **c** involves subtractive cancellation at some other point, describe how to avoid that difficulty.

16. Let $f(x) = -e^{-2x} + e^x$. Which of these formulas x, $3x$, $3x(1 - x/2)$, $2 - 3x$, or $e^x(1 - e^{-3x})$ for f is most accurate for small values of x?

17. If at most 2 bits of precision are to be lost in the computation $y = \sqrt{x^2 + 1} - 1$, what restriction must be placed on x?

18. For what range of values of θ does the approximation $\sin\theta \approx \theta$ give results correct to three (rounded) decimal places?

19. For small values of x, how good is the approximation $\cos x \approx 1$? How small must x be to have $\frac{1}{2} \times 10^{-8}$ accuracy?

20. The series $\sum_{k=1}^{\infty} k^{-1}$ is called the **harmonic series**. It diverges. The partial sums, $S_n = \sum_{k=1}^{n} k^{-1}$, can be computed recursively by setting $S_1 = 1$ and using $S_n = S_{n-1} + n^{-1}$. If this computation were carried out on your computer, what is the largest S_n that would be obtained? (Do *not* do this experimentally on the computer; it is too expensive.) See Schechter [1984].

4. The condition number of the function $f(x) = x^\alpha$ is independent of x. What is the condition number?

5. What are the condition numbers of the following functions? Where are they large?

   a. $(x-1)^\alpha$

   b. $\ln x$

   c. $\sin x$

   d. e^x

   e. $x^{-1}e^x$

   f. $\cos^{-1} x$

6. cpb2.3.6 Consider the example in the text for which $y_{n+1} = e - (n+1)y_n$. How many decimal places of accuracy should be used in computing $y_1, y_2, \ldots, y_{20}$ if y_{20} is to be accurate to five decimal places?

7. Show that the recurrence relation

$$x_n = 2x_{n-1} + x_{n-2}$$

has a general solution of the form

$$x_n = A\lambda^n + B\mu^n$$

Is the recurrence relation a good way to compute x_n from arbitrary initial values x_0 and x_1?

8. The **Fibonacci sequence** is generated by the formulas

$$\begin{cases} r_0 = 1 & r_1 = 1 \\ r_{n+1} = r_n + r_{n-1} \end{cases}$$

The sequence therefore starts out 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Prove that the sequence $[2r_n/r_{n-1}]$ converges to $1+\sqrt{5}$. Is the convergence linear, superlinear, quadratic?

9. (Continuation) If the recurrence relation in the preceding problem is used with starting values $r_0 = 1$ and $r_1 = (1-\sqrt{5})/2$, what is the theoretically correct value of r_n $(n \geq 2)$? Can the recurrence relation provide a stable means for computing r_n in this case?

COMPUTER PROBLEMS 2.3

1. Let sequences $[A_n]$ and $[B_n]$ be generated as follows:

$$\begin{cases} A_0 = 0 & A_1 = 1 \\ A_n = nA_{n-1} + A_{n-2} \end{cases} \qquad \begin{cases} B_0 = 1 & B_1 = 1 \\ B_n = nB_{n-1} + B_{n-2} \end{cases}$$

What is $\lim_{n\to\infty}(A_n/B_n)$?

2. The **Bessel functions** Y_n satisfy the same recurrence formula that the functions J_n satisfy. (See Section 1.3, p. 34.) However, they use *different starting values*. For $x = 1$, they are

$$Y_0(1) = 0.08825\,69642 \qquad Y_1(1) = -0.78121\,28213$$

Compute $Y_2(1), Y_3(1), \ldots, Y_{20}(1)$ using the recurrence formula. Try to decide whether the results are reliable. *Hint:* The numbers $|Y_n(1)|$ should grow rapidly. Perhaps you can prove an inequality, such as $|Y_n(1)/Y_{n-1}(1)| > n$.