6.8

6. $a_{ij} = \int_0^1 x^i x^j dx = \int_0^1 x^{i+j} dx = \frac{1}{i+j+1} \quad for \quad 0 \le i, j \le n-1$

7. In the Gram-Schmidt process, u_j is the linear combination of $\{v_1, v_2, ..., v_j\}$. That is $a_{ij} = 0$ for $i > j$. Hence the coefficent matix is upper triangular.

21. Just continue to follow the calculation in Example 2.

22. Given Inner product is $< f, g >= \int_0^1 f(x)g(x)dx$.

$$p_0 = 1, \quad a_1 = \frac{< xp_0, p_0 >}{< p_0, p_0 >} = \frac{\int_0^1 xdx}{\int_0^1 1dx}$$

$$p_1 = x - \frac{1}{2}$$

$$a_2 = \frac{< xp_1, p_1 >}{< p_1, p_1 >} = \frac{\int_0^1 x(x-\frac{1}{2})^2 dx}{\int_0^1 (x-\frac{1}{2})^2 dx} = \frac{1}{2}, \quad b_2 = \frac{< xp_1, p_0 >}{< p_0, p_0 >} = \frac{\int_0^1 x(x-\frac{1}{2})dx}{\int_0^1 1dx} = \frac{1}{12}$$

$$p_2 = (x - \frac{1}{2})(x - \frac{1}{2}) - \frac{1}{12} = x^2 - x + \frac{1}{6}$$

$$a_3 = \frac{< xp_2, p_2 >}{< p_2, p_2 >} = \frac{1}{2}, \quad b_3 = \frac{< xp_2, p_1 >}{< p_1, p_1 >} = \frac{1}{15}$$

$$p_3 = (x - \frac{1}{2})(x^2 - x + \frac{1}{6}) - \frac{1}{15}(x - \frac{1}{2}) = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}$$

6.12

1. By assumption,
$g(x_j) = \sum_{k=0}^{N-1} a_k E_k(x_j) = 0 \Longrightarrow g(x_j)\overline{E_l(x_j)} = \sum_{k=0}^{N-1} a_k E_k(x_j)\overline{E_l(x_j)} = 0$ for $o \le l \le N-1$
Hence,
$0 = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} a_k E_k(x_j)\overline{E_l(x_j)} = \sum_{k=0}^{N-1} a_k \sum_{j=0}^{N-1} E_k(x_j)\overline{E_l(x_j)} = \sum_{k=0}^{N-1} a_k N < E_k, E_l >_N = Na_l$
So $a_l = 0$ for $o \le l \le N-1$

4. $n < f, E_{-j} >_n = \sum_{k=0}^{n-1} f(x_k)\overline{E_{-j}}(x_k) = \sum_{k=0}^{n-1} f(x_k)E_j(x_k) = \sum_{k=0}^{n-1} < g, E_k >_n E_j(x_k)$
$= \sum_{k=0}^{n-1} < g, E_k >_n E_k(x_j) = g(x_j)$

5. By Theorem 2, $< E_k, E_0 >_n = \begin{cases} 1 & \text{if n devides k;} \\ 0 & \text{Otherwise.} \end{cases}$

But $< E_k, E_0 >_n = \frac{1}{n}\sum_{j=0}^{n-1} e^{\frac{2\pi ikj}{n}} = \frac{1}{n}\sum_{j=0}^{n-1} \cos(\frac{2\pi kj}{n}) + i\sin(\frac{2\pi kj}{n})$

Hence,

$$\frac{1}{n}\sum_{j=0}^{n-1} \cos(\frac{2\pi kj}{n}) = \begin{cases} 1 & \text{if n devides k;} \\ 0 & \text{Otherwise.} \end{cases}$$

$$\frac{1}{n}\sum_{j=0}^{n-1} \sin(\frac{2\pi kj}{n}) = 0$$