Lecture Note I: Mathematical Models of Continuum Mechanics

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In this chapter, we shall introduce mathematical models for Darcy's flow in porous media, elastic equations of solids, for incompressible Newtonian fluid flow, and Maxwell's equations in electromagnetic.

Let Ω be a bounded, open, connected subset of $\Re^d$ ($d = 2$ or 3) with a Lipschitz continuous boundary $\partial\Omega$. Denote $\mathbf{n} = (n_1, \dots, n_d)$ the outward unit vector normal to the boundary. We partition the boundary of the domain Ω into two open subsets Γ_D and Γ_N such that $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$.

0.1 Notation

For a scalar function $f(\mathbf{x})$ with $\mathbf{x} = (x_1, \dots, x_d) \in \Omega$, define its gradient as a row vector

$$\nabla f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_d}).$$

For a vector function $\mathbf{v} = (v_1, \dots, v_d)$, define its divergence by

$$\nabla \cdot \mathbf{v} = \sum_{i=1}^{d} \frac{\partial v_i}{\partial x_i}$$

and its curl for $d = 3$ by

$$\nabla\times\mathbf{v} = (\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}).$$

For $d = 2$, by identifying $\Re^2$ with the (x_1, x_2)-plane in $\Re^3$, the curl of $\mathbf{v} = (v_1, v_2)$ means the scalar function

$$\nabla\times\mathbf{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

For convenience, we define its formal adjoint by

$$\nabla^{\perp} f = (\frac{\partial f}{\partial x_2}, -\frac{\partial f}{\partial x_1}).$$

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For a vector function $\mathbf{v} = (v_1, \dots, v_d)$, define its gradient as a $d \times d$ tensor

$$\nabla \mathbf{v} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \cdots & \frac{\partial v_1}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial v_d}{\partial x_1} & \cdots & \frac{\partial v_d}{\partial x_d} \end{pmatrix} = \left(\frac{\partial v_i}{\partial x_j} \right)_{d \times d}.$$

For a tensor function $\boldsymbol{\tau} = (\tau_{ij})_{d \times d}$, let $\boldsymbol{\tau}_i = (\tau_{i1}, \dots, \tau_{id})$ denote its i^{th}-row for $i = 1, \dots, d$, and define its divergence, normal, and trace by

$$\nabla \cdot \boldsymbol{\tau} = (\nabla \cdot \boldsymbol{\tau}_1, \dots, \nabla \cdot \boldsymbol{\tau}_d), \quad \mathbf{n} \cdot \boldsymbol{\tau} = (\mathbf{n} \cdot \boldsymbol{\tau}_1, \dots, \mathbf{n} \cdot \boldsymbol{\tau}_d), \quad \text{and} \quad \operatorname{tr} \boldsymbol{\tau} = \sum_{i=1}^{d} \tau_{ii},$$

respectively. Finally, denote the Laplacian of scalar and vector functions by

$$\Delta f = \nabla \cdot (\nabla f) = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2} \quad \text{and} \quad \Delta \mathbf{v} = (\Delta v_1, \dots, \Delta v_d)$$

respectively. Below we list some useful identities.

$$-\Delta \mathbf{v} + \nabla(\nabla \cdot \mathbf{v}) = \begin{cases} \nabla \times (\nabla \times \mathbf{v}), & d = 3, \\ \nabla^{\perp}(\nabla \times \mathbf{v}), & d = 2, \end{cases} \tag{0.1}$$

$$\nabla \cdot (\nabla \mathbf{v})^t = \nabla(\nabla \cdot \mathbf{v}), \tag{0.2}$$

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0, \quad \text{and} \quad \nabla \times (\nabla \mathbf{v}) = \mathbf{0}. \tag{0.3}$$

1 Darcy's Flow in Porous Media

Let f be a given source/sink term, k (scalar or anisotropic tensor) the permeability, and μ the fluid viscosity. Darcy's flow in porous media occupying a region Ω consists in finding a velocity vector field $\mathbf{u} = (u_1, \dots, u_d)$ and a pressure p that satisfy the conservation of mass:

$$\nabla \cdot \mathbf{u} = f \quad \text{in } \Omega, \tag{1.4}$$

Darcy's law (the constitutive equation):

$$\mathbf{u} = -\frac{k}{\mu} \nabla p \quad \text{in } \Omega, \tag{1.5}$$

and boundary conditions:

$$p = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n} \cdot \mathbf{u} = 0 \quad \text{on } \Gamma_N. \tag{1.6}$$

For simplicity, here we assume that the boundary conditions are homogeneous.

2 Elastic Equations

Let $\mathbf{f} = (f_1, \ldots, f_d)$ be a given body force and $\mathbf{g} = (g_1, \ldots, g_d)$ be the given traction. Denote by ρ the density, $\mathbf{u} = (u_1, \ldots, u_d)$ the displacement field, and $\boldsymbol{\sigma} = \left(\sigma_{ij}\right)_{d\times d}$ the stress tensor. In a Lagrangian frame, motion of the elastic body is governed by the conservation of mass:

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{u} = 0, \tag{2.7}$$

and by the balance of linear momentum:

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \nabla \cdot \boldsymbol{\sigma} = \rho \mathbf{f}. \tag{2.8}$$

The above system is closed by using the constitutive equation that expresses a relation between the stress and the strain tensors:

$$\boldsymbol{\sigma} = \mathcal{C}_\lambda \boldsymbol{\epsilon}(\mathbf{u}) \quad \text{or} \quad \boldsymbol{\epsilon}(\mathbf{u}) = \mathcal{A}_\lambda \boldsymbol{\sigma} \tag{2.9}$$

where $\mathcal{C}_\lambda$ and $\mathcal{A}_\lambda = \mathcal{C}_\lambda^{-1}$ are the elasticity and the compliance tensors of fourth-order, respectively. Define the linearized strain tensor by

$$\boldsymbol{\epsilon}(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^t\right) = \left(\epsilon_{ij}(\mathbf{u})\right)_{d\times d} \quad \text{with} \quad \epsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

For an isotropic elastic material, the elasticity and compliance tensors have the following simple expressions:

$$\mathcal{C}_\lambda \boldsymbol{\epsilon}(\mathbf{u}) = \lambda \left(\operatorname{tr} \boldsymbol{\epsilon}(\mathbf{u})\right)\boldsymbol{\delta} + 2\mu \boldsymbol{\epsilon}(\mathbf{u}) \tag{2.10}$$

and

$$\mathcal{A}_\lambda \boldsymbol{\sigma} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{d\lambda + 2\mu}(\operatorname{tr} \boldsymbol{\sigma})\boldsymbol{\delta}\right), \tag{2.11}$$

where $\boldsymbol{\delta} = (\delta_{ij})_{d\times d}$ is the identity tensor, and positive constants λ and μ are the Lamé constants such that $\mu \in [\mu_1, \mu_2]$ with $0 < \mu_1 < \mu_2$ and $\lambda \in (\alpha, \infty)$ with $\alpha > 0$. Materials are said to be nearly incompressible or incompressible when λ is very large or infinite, respectively.

When the elastic body reaches equilibrium state, we then have the following stress-displacement system:

$$\left\{ \begin{array}{rcll} \mathcal{A}_\lambda \boldsymbol{\sigma} - \boldsymbol{\epsilon}(\mathbf{u}) & = & \mathbf{0} & \text{in} \quad \Omega, \\ -\nabla \cdot \boldsymbol{\sigma} & = & \mathbf{f} & \text{in} \quad \Omega, \end{array} \right. \tag{2.12}$$

where $\mathbf{f}$ is scaled by multiplying ρ. In general, we have the following clamped and traction boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{g} \quad \text{on } \Gamma_N. \tag{2.13}$$

A simple calculation gives that

$$\nabla \cdot (\mathcal{C}_\lambda \, \boldsymbol{\epsilon}(\mathbf{u})) = \mu \, \Delta \mathbf{u} + (\lambda + \mu) \nabla \, (\nabla \cdot \mathbf{u})$$

and that

$$\mathbf{n} \cdot (\mathcal{C}_\lambda \, \boldsymbol{\epsilon}(\mathbf{u})) = \lambda \left(\nabla \cdot \mathbf{u} \right) \mathbf{n} + 2\mu \, \mathbf{n} \cdot \boldsymbol{\epsilon}(\mathbf{u}).$$

Now, eliminating the stress in (2.12), we then get the displacement formulation:

$$\left\{ \begin{array}{rcll} -\mu \, \Delta \mathbf{u} - (\lambda + \mu) \nabla \, (\nabla \cdot \mathbf{u}) & = & \mathbf{f} & \text{in} \quad \Omega, \\ \mathbf{u} & = & \mathbf{0} & \text{on} \quad \Gamma_D, \\ \lambda \left(\nabla \cdot \mathbf{u} \right) \mathbf{n} + 2\mu \, \mathbf{n} \cdot \boldsymbol{\epsilon}(\mathbf{u}) & = & \mathbf{g} & \text{on} \quad \Gamma_N. \end{array} \right. \tag{2.14}$$

3 Incompressible Newtonian Flow

Let $\mathbf{f} = (f_1, \ldots, f_d)$ be a given external body force defined in a domain Ω and $\mathbf{g} = (g_1, \ldots, g_d)$ be the given traction. Let $\mathbf{u}(\mathbf{x}, t) = (u_1, \ldots, u_d)$ be the velocity vector field of a particle of fluid that is moving through $\mathbf{x}$ at time t, and let $\boldsymbol{\sigma} = (\sigma_{ij})_{d \times d}$ be the stress tensor field. In an Eulerian frame, motion of fluids is governed by the conservation of mass:

$$\frac{D\,\rho}{D\,t} + \rho \, \nabla \cdot \mathbf{u} = 0, \tag{3.15}$$

and by the balance of linear momentum:

$$\rho \, \frac{D\,\mathbf{u}}{D\,t} - \nabla \cdot \boldsymbol{\sigma} = \rho \, \mathbf{f}, \tag{3.16}$$

where the material derivative is defined by

$$\frac{D}{D\,t} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla = \frac{\partial}{\partial t} + \sum_{i=1}^{d} u_i \frac{\partial}{\partial x_i}.$$

For Newtonian fluids, one has the following constitutive equation

$$\boldsymbol{\sigma} = -p \, \boldsymbol{\delta} + 2\mu \left(\boldsymbol{\epsilon}(\mathbf{u}) - \frac{1}{d} (\mathrm{tr} \, \boldsymbol{\epsilon}(\mathbf{u})) \boldsymbol{\delta} \right), \tag{3.17}$$

where p is the hydrostatic pressure, μ is the viscosity parameter and

$$\boldsymbol{\epsilon}(\mathbf{u}) = \frac{1}{2} \left(\nabla \, \mathbf{u} + (\nabla \, \mathbf{u})^t \right)$$

is the deformation rate tensor. When fluids are incompressible, then the density is constant. Hence, the conservation of mass in (2.7) implies the incompressibility condition

$$\nabla \cdot \mathbf{u} = 0. \tag{3.18}$$

Scale $\mathbf{u}$ and $\mathbf{f}$ by multiplying ρ and let $\nu = \mu/\rho$, we then get the stress-velocity-pressure formulation for incompressible Newtonian fluid flow:

$$\left\{ \begin{array}{rcll} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot \boldsymbol{\sigma} & = & \mathbf{f} & \text{in} \quad \Omega \times (0,\, T), \\ \boldsymbol{\sigma} + p\,\boldsymbol{\delta} - 2\nu\,\boldsymbol{\epsilon}(\mathbf{u}) & = & \mathbf{0} & \text{in} \quad \Omega \times (0,\, T), \\ \nabla \cdot \mathbf{u} & = & 0 & \text{in} \quad \Omega \times (0,\, T). \end{array} \right. \tag{3.19}$$

System (3.19) is supplemented with the following boundary and initial conditions

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega \times (0,\, T) \quad \text{and} \quad \mathbf{u}(\mathbf{x},\, 0) = \mathbf{v}^0 \quad \text{in } \Omega \tag{3.20}$$

where $\mathbf{g} = (g_1,\, \cdots,\, g_d)$ is given and satisfies the compatibility condition

$$\int_{\partial\Omega} \mathbf{n} \cdot \mathbf{g}\, ds = 0$$

and $\mathbf{u}^0$ is given initial velocity. Using identity (0.2) and the incompressibility condition in (3.18) give that

$$2\,\nabla \cdot \boldsymbol{\epsilon}(\mathbf{u}) = \Delta\,\mathbf{u}.$$

Now, differentiating and eliminating the stress in the above system leads to the well-known incompressible Naiver-Stokes equations:

$$\left\{ \begin{array}{rcll} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{u} - \nu\,\Delta\,\mathbf{u} + \nabla\,p & = & \mathbf{f} & \text{in} \quad \Omega \times (0,\, T), \\ \nabla \cdot \mathbf{u} & = & 0 & \text{in} \quad \Omega \times (0,\, T) \end{array} \right. \tag{3.21}$$

with the boundary and initial conditions in (3.20).

In (3.19), taking the trace of the second equation and using the third equation we have that

$$p = -\frac{1}{d}\,\mathrm{tr}\,\boldsymbol{\sigma}, \tag{3.22}$$

which is commonly used in continuum mechanics to define the *hydrostatic* pressure. Using (3.22) we eliminate the pressure in the second equation of (3.19) to obtain the following constitutive equation:

$$\mathcal{A}\,\boldsymbol{\sigma} = \boldsymbol{\epsilon}(\mathbf{u}) \quad \text{in} \quad \Omega \times (0,\, T), \tag{3.23}$$

where $\mathcal{A}\,\boldsymbol{\sigma}$ is the scaled deviatoric stress:

$$\mathcal{A}\,\boldsymbol{\sigma} = \frac{1}{2\nu}\left(\boldsymbol{\sigma} - \frac{1}{d}\,(\mathrm{tr}\,\boldsymbol{\sigma})\,\boldsymbol{\delta}\right).$$

Note that the trace of this equation yields the incompressibility condition in (3.18). This and the momentum equation define the stress-velocity formulation for incompressible Newtonian fluid flow problems:

$$\left\{ \begin{array}{rcll} \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot \boldsymbol{\sigma} & = & \mathbf{f} & \text{in} \quad \Omega \times (0,\, T) \\ \mathcal{A}\,\boldsymbol{\sigma} - \boldsymbol{\epsilon}(\mathbf{u}) & = & \mathbf{0} & \text{in} \quad \Omega \times (0,\, T) \end{array} \right. \tag{3.24}$$

with the boundary and initial conditions in (3.20).

4 Maxwell's Equations

Maxwell's equations consist of a set of fundamental equations governing macroscopic electromagnetic phenomena. For general time-varying fields, the original Maxwell's equations in a "stable" medium can be written as

$$\left\{ \begin{array}{rcll} \dfrac{\partial(\epsilon \mathbf{E})}{\partial t} + \sigma \mathbf{E} - \nabla\times\mathbf{H} & = & \mathbf{J} & \text{in } \Omega \times (0,\, T), \\ \dfrac{\partial(\mu \mathbf{H})}{\partial t} + \nabla\times\mathbf{E} & = & \mathbf{0} & \text{in } \Omega \times (0,\, T), \\ \nabla \cdot (\epsilon \mathbf{E}) & = & q & \text{in } \Omega \times (0,\, T), \\ \nabla \cdot (\mu \mathbf{H}) & = & 0 & \text{in } \Omega \times (0,\, T), \end{array} \right. \tag{4.25}$$

where $\Omega \subset \Re^3$. Here, we shall assume that Ω is a bounded and simply connected domain with a piecewise smooth boundary; below, we allow Γ to be decomposed into $\Gamma = \Gamma_1 \cup \Gamma_2$, where either Γ_1 or Γ_2 can be empty. The electric and magnetic fields are given by $\mathbf{E}$ and $\mathbf{H}$, respectively. The constitutive tensors ϵ, μ, and σ denote the dielectric permittivity, magnetic permeability, and conductivity, respectively, of the medium; $\mathbf{J}$ and q are given functions specifying the applied current and charge. The symbols $\nabla\times$ and $\nabla\cdot$ denote the curl and divergence operators.

Boundary conditions are given by

$$\mathbf{n} \times \mathbf{E} = \mathbf{0} \quad \text{and} \quad \mathbf{n} \cdot (\mu \mathbf{H}) = 0 \quad \text{on } \Gamma_1 \times (0,\, T) \tag{4.26}$$

for a perfectly conducting wall Γ_1 and

$$\mathbf{n} \times \mathbf{H} = \mathbf{0} \quad \text{and} \quad \mathbf{n} \cdot (\epsilon \mathbf{E}) = 0 \quad \text{on } \Gamma_2 \times (0,\, T) \tag{4.27}$$

for a perfectly magnetic wall Γ_2, where $\mathbf{n}$ is the outward unit normal to the boundary Γ. Initial conditions are specified by

$$\mathbf{E}(x,\, 0) = \mathbf{E}_0(x) \quad \text{and} \quad \mathbf{H}(x,\, 0) = \mathbf{H}_0(x) \quad \text{in } \Omega, \tag{4.28}$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are the initial electric and magnetic fields satisfying

$$\nabla \cdot (\mu \mathbf{H}_0) = 0 \quad \text{in } \Omega \quad \text{and} \quad \mathbf{n} \cdot (\mu \mathbf{H}_0) = 0 \quad \text{on } \Gamma_1.$$

Lecture Note II: Variational Problems for Elastic Equations

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In this chapter, we shall introduce elementary properties for some functional spaces and abstract theory for variational problems, and establish well-posedness of elastic equations.

1 Sobolev Spaces

Assume that the boundary $\partial\Omega$ of the domain $\Omega \subset \mathcal{R}^d$ is sufficiently smooth (for instance a Lipschitz continuous boundary). Denote the space of square integrable functions on Ω by

$$L^2(\Omega) = \{v \mid \int_\Omega |v|^2\,dx = \|v\|_{0,\Omega}^2 < +\infty\}.$$

Let

$$\alpha = (\alpha_1,\,...,\,\alpha_d) \quad \text{with} \quad |\alpha| = \sum_{i=1}^d \alpha_i,$$

be multiple index and denote the $|\alpha|$-th derivative of v in the sense of distributions by

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}.$$

For integer $m \ge 0$, define Sobolev spaces by

$$H^m(\Omega) = \{v \mid D^\alpha v \in L^2(\Omega)\ \forall\,|\alpha| \le m\},$$

with the semi-norm

$$|v|_{m,\Omega} = \left(\sum_{|\alpha|=m} \|D^\alpha v\|_{0,\Omega}^2\right)^{\frac{1}{2}}$$

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and the norm

$$\|v\|_{m,\Omega} = \left(\sum_{|\alpha| \le m} \|D^\alpha v\|_{0,\Omega}^2 \right)^{\frac{1}{2}}.$$

Denote by $\mathcal{D}(\Omega)$ or $\mathcal{C}_0^\infty(\Omega)$ the space of infinitely differentiable functions having a compact support in Ω and

$$\mathcal{D}(\bar{\Omega}) = \{\varphi|_\Omega \,:\, \varphi \in \mathcal{D}(\mathcal{O}) \text{ for some open subset } \Omega \subset \mathcal{O}/ \subset \Re^d\}.$$

Let $H_0^m(\Omega)$ be the closure of $\mathcal{D}(\Omega)$ in the $\|\cdot\|_{m,\Omega}$. We shall denote by $L_0^2(\Omega)$ the subspace of $L^2(\Omega)$ having zero mean value.

We shall need to consider functions that vanish on a part of the boundary. We partition the boundary of the domain Ω into two open subsets Γ_D and Γ_N such that $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. One defines a subspace of $H^1(\Omega)$ by

$$H_{0,D}^1(\Omega) = \{v \in H^1(\Omega) \,|\, v = 0 \text{ on } \Gamma_D\}.$$

When $\Gamma_D = \partial\Omega$, then $H_{0,D}^1(\Omega) = H_0^1(\Omega)$; when $\Gamma_D = \emptyset$, then $H_{0,D}^1(\Omega) = H^1(\Omega)$. We use $H_{0,D}^{-1}(\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$ to denote the dual of $H_{0,D}^1(\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$ with standard dual norms:

$$\|\phi\|_{-1,D} = \sup_{0 \ne \psi \in H_{0,D}^1(\Omega)} \frac{(\phi,\, \psi)}{\|\psi\|_1}, \quad \text{and} \quad \|\phi\|_{-1/2,\partial\Omega} = \sup_{0 \ne \psi \in H^{\frac{1}{2}}(\partial\Omega)} \frac{< \phi,\, \psi >_{0,\partial\Omega}}{\|\psi\|_{1/2,\partial\Omega}},$$

respectively,

Theorem 1.1 *Suppose that the domain Ω has a Lipschitz boundary and that $p \in [1, \infty]$ is a real number. Then*

$$\|v\|_{0,\partial\Omega} \le C\, \|v\|_{0,\Omega}^{\frac{1}{2}}\, \|v\|_{1,\Omega}^{\frac{1}{2}}. \tag{1.1}$$

1.1 Poincaré Inequality

Consider the following two subspaces of $H^1(\Omega)$:

$$H_{0,D}^1(\Omega) = \{v \in H^1(\Omega) \,|\, v|_{\Gamma_D} = 0\} \quad \text{and} \quad \hat{H}^1(\Omega) = \{v \in H^1(\Omega) \,|\, \int_\Omega v\, dx = 0\}.$$

We have the following **Poincaré Inequality**: there exists a positive constant C such that

$$\|v\|_{0,\Omega} \le C\, |v|_{1,\Omega} \tag{1.2}$$

for either all $v \in H_{0,D}^1(\Omega)$ when $\text{mes}\,(\Gamma_D) \ne 0$ or all $v \in \hat{H}^1(\Omega)$.

1.2 Integration Formulas

Denote the product space $L^2(\Omega)^d = \prod_{i=1}^{d} L^2(\Omega)$ with the standard product norm

$$\|\mathbf{v}\|_{0,\Omega} = \left(\sum_{i=1}^{d} \|v_i\|_{0,\Omega}^2 \right)^{\frac{1}{2}}.$$

Then, set

$$H(\text{div}; \Omega) = \{\mathbf{v} \in L^2(\Omega)^d \,|\, \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$$

which is a Hilbert space under the norm

$$\|\mathbf{v}\|_{0,\Omega} = \left(\|\mathbf{v}\|_{0,\Omega}^2 + \|\nabla \cdot \mathbf{v}\|_{0,\Omega}^2\right)^{\frac{1}{2}}.$$

It is easy to check the following integration formulas:

$$\int_\Omega \nabla \cdot \mathbf{v}\, dx = \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{v}\, ds \qquad \forall\ \mathbf{v} \in H(\text{div}; \Omega);$$

$$\int_\Omega \frac{\partial v}{\partial x_i} w\, dx = -\int_\Omega v \frac{\partial w}{\partial x_i}\, dx + \int_{\partial\Omega} v w n_i\, ds \qquad \forall\ v,\ w \in H^1(\Omega);$$

$$\int_\Omega (\nabla \cdot \mathbf{v}) w\, dx = -\int_\Omega \mathbf{v} \cdot \nabla w\, dx + \int_{\partial\Omega} (\mathbf{n} \cdot \mathbf{v}) w\, ds \qquad \forall\ \mathbf{v} \in H^1(\Omega)^d, \forall\ w \in H^1(\Omega).$$

2 The Lax-Milgram Theorem

A bilinear form, $a(\cdot, \cdot)$, on a linear space V is a mapping $a : V \times V \longrightarrow \mathcal{R}$ such that

$$a(\alpha u + \beta v, w) = \alpha\, a(u, w) + \beta\, a(v, w) \quad \text{and} \quad a(u, \alpha v + \beta w) = \alpha\, a(u, v) + \beta\, a(u, w)$$

for all $u, v, w \in V$ and $\alpha, \beta \in \mathcal{R}$. It is symmetric if $a(u, v) = a(v, u)$ for all $u, v \in V$.

Assume that $(V, (\cdot, \cdot)_V)$ is an inner-product space, one then has the following **Schwarz Inequality**

$$|(u, v)_V| \le (u, u)_V^{\frac{1}{2}} (v, v)_V^{\frac{1}{2}}. \tag{2.3}$$

The equality holds if and only if u and v are linearly dependent.

A bilinear form $a(\cdot, \cdot)$ on a normed linear space $(H, \|\cdot\|_H)$ is continuous if there exists a constant C such that

$$|a(u, v)| \le C\, \|u\|_H\, \|v\|_H \quad \forall\ u, v \in H$$

and coercive on a subspace $H_1 \subset H$ if there exists $\alpha > 0$ such that

$$a(v, v) \ge \alpha\, \|v\|_H^2 \quad \forall\ v \in H_1.$$

Theorem 2.1 (Lax-Milgram Theorem) *Given a Hilbert space* $(V, (\cdot, \cdot)_V)$, *assume that* $a(\cdot, \cdot) : V \times V \to \mathcal{R}$ *is bilinear, continuous, and coercive and that* $f(\cdot) : V \to \mathcal{R}$ *is linear and continuous. Then there exists a unique solution* $u \in V$ *such that*

$$a(u, v) = f(v) \quad \forall\, v \in V.$$

Homework: prove the Lax-Milgram theorem.

3 Variational Formulation for Elastic Equations

Consider the following elastic equation:

$$\begin{cases} \mathcal{A}_\lambda \boldsymbol{\sigma} - \boldsymbol{\epsilon}(\mathbf{u}) & = \mathbf{0} \quad \text{in} \quad \Omega, \\ -\nabla \cdot \boldsymbol{\sigma} & = \mathbf{f} \quad \text{in} \quad \Omega, \end{cases} \tag{3.4}$$

with boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n} \cdot \boldsymbol{\sigma} = \mathbf{g} \quad \text{on } \Gamma_N. \tag{3.5}$$

To derive variational formulation, let us first define multiplication of two tensors $A = (a_{ij})_{d\times d}$ and $B = (b_{ij})_{d\times d}$:

$$A : B = \sum_{i,j=1}^{d} a_{ij}\, b_{ij}.$$

It is easy to check that if A is symmetric and B is skew-symmetric ($B^t = -B$), then

$$A : B = 0. \tag{3.6}$$

In particular, since $\nabla\, \mathbf{v}$ may be decomposed into its symmetric and skew-symmetric parts

$$\nabla\, \mathbf{v} = \boldsymbol{\epsilon}(\mathbf{v}) + \frac{1}{2}\left(\nabla\, \mathbf{v} - (\nabla\, \mathbf{v})^t\right)$$

we then have that for symmetric $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} : \nabla\, \mathbf{v} = \boldsymbol{\sigma} : \boldsymbol{\epsilon}(\mathbf{v}).$$

Now, multiplying the second equation of (3.4) by a test vector function $\mathbf{v}$, integrating over the domain Ω, and using integration by parts and the traction boundary condition in (3.5) give

$$\begin{aligned} (\mathbf{f}, \mathbf{v}) &= (-\nabla \cdot \boldsymbol{\sigma}, \mathbf{v}) = (\boldsymbol{\sigma}, \nabla\, \mathbf{v}) - \int_{\partial\Omega} (\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \mathbf{v}\, ds \\ &= (\boldsymbol{\sigma}, \boldsymbol{\epsilon}(\mathbf{v})) - \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v}\, ds - \int_{\Gamma_D} (\mathbf{n} \cdot \boldsymbol{\sigma}) \cdot \mathbf{v}\, ds. \end{aligned}$$

If $\text{mes}(\Gamma_D) \neq 0$, then the variational form is to find $\mathbf{u} \in H^1_{0,D}(\Omega)^d$ such that

$$a(\mathbf{u},\, \mathbf{v}) = f(\mathbf{v}) \quad \forall\, \mathbf{v} \in H^1_{0,D}(\Omega)^d \tag{3.7}$$

where the bilinear form is given by

$$a(\mathbf{u},\, \mathbf{v}) = (\mathcal{C}_\lambda \boldsymbol{\epsilon}(\mathbf{u}),\, \boldsymbol{\epsilon}(\mathbf{v})) = 2\mu\, (\boldsymbol{\epsilon}(\mathbf{u}),\, \boldsymbol{\epsilon}(\mathbf{v})) + \lambda\, (\nabla\cdot\mathbf{u},\, \nabla\cdot\mathbf{v})$$

and the linear form is given by

$$f(\mathbf{v}) = (\mathbf{f},\, \mathbf{v}) + \int_{\Gamma_N} \mathbf{g}\cdot\mathbf{v}\, ds.$$

In the remaining of this section, we restrict ourself in two dimensions because of proofs of Korn's inequalities. For the pure traction problem, i.e., $\Gamma_N = \partial\Omega$, the elastic equations have many solutions in $H^1(\Omega)^d$. To find a unique solution, we introduce the space of infinitesimal rigid motions (i.e., the kernel of $\boldsymbol{\epsilon}$) in $\mathcal{R}^2$:

$$\text{RM} = \{\mathbf{v} \,|\, \mathbf{v} = (a,\, b) + c\,(x_2,\, -x_1),\ \ a,\, b,\, c \in \mathcal{R}\}.$$

Homework: define RM in $\mathcal{R}^3$.

Define

$$\hat{H}(\Omega)^2 = \{\mathbf{v} \in H^1(\Omega)^2 \,|\, \int_\Omega \mathbf{v}\, dx = \mathbf{0} \text{ and } \int_\Omega \nabla\times\mathbf{v}\, dx = 0\},$$

then

$$H^1(\Omega)^2 = \hat{H}^1(\Omega)^2 \oplus \text{RM}.$$

In particular, for any given $\mathbf{v} \in H^1(\Omega)^2$, there exists a unique pair $(\mathbf{z},\, \mathbf{w}) \in \hat{H}^1(\Omega)^2 \times \text{RM}$ such that

$$\mathbf{v} = \mathbf{z} + \mathbf{w} \quad \text{with} \quad \mathbf{w} = (a,\, b) + c\,(x_2,\, -x_1) \tag{3.8}$$

where

$$c = -\frac{1}{2|\Omega|}\int_\Omega \nabla\times\mathbf{v}\, dx \quad \text{and} \quad (a,\, b) = \frac{1}{|\Omega|}\int_\Omega \big(\mathbf{v} - c\,(x_2,\, -x_1)\big)\, dx. \tag{3.9}$$

Since the domain is bounded, it is then easy to show, by using (3.9), that

$$\|\mathbf{w}\|_{1,\Omega} \le C\, \|\mathbf{v}\|_{1,\Omega}$$

and, by the triangle inequality, that

$$\|\mathbf{z}\|_{1,\Omega} = \|\mathbf{v} - \mathbf{w}\|_{1,\Omega} \le \|\mathbf{v}\|_{1,\Omega} + \|\mathbf{w}\|_{1,\Omega} \le C\, \|\mathbf{v}\|_{1,\Omega}.$$

Hence,

$$\|\mathbf{z}\|_{1,\Omega} + \|\mathbf{w}\|_{1,\Omega} \le C\, \|\mathbf{v}\|_{1,\Omega}. \tag{3.10}$$

Homework: derive (3.9) and prove (3.10).

Now, the variational problem is to find $\mathbf{u} \in \hat{H}^1(\Omega)^d$ such that

$$a(\mathbf{u}, \mathbf{v}) = f(\mathbf{v}) \quad \forall\, \mathbf{v} \in \hat{H}^1(\Omega)^d \tag{3.11}$$

with the linear form given by

$$f(\mathbf{v}) = (\mathbf{f}, \mathbf{v}) + \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v}\, ds.$$

that satisfies the following compatibility condition

$$f(\mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in \mathrm{RM}. \tag{3.12}$$

To establish the well-posedness of variational problems (3.7) and (3.11) by using the Lax-Milgram theorem, main task is to show the coercivity of the bilinear form in appropriate spaces. To do so, we need the following fundamental inequalities, called Korn's inequalities.

Theorem 3.1 (Second Korn Inequality) *There exists a positive constant C such that*

$$\|\mathbf{v}\|_{1,\Omega} \le C\, \|\boldsymbol{\epsilon}(\mathbf{v})\|_{0,\Omega} \quad \forall\, \mathbf{v} \in \hat{H}^1(\Omega)^2. \tag{3.13}$$

Theorem 3.2 (Korn's Inequality) *There exists a positive constant C such that*

$$\|\mathbf{v}\|_{1,\Omega} \le C\, \left(\|\mathbf{v}\|_{0,\Omega} + \|\boldsymbol{\epsilon}(\mathbf{v})\|_{0,\Omega}\right) \quad \forall\, \mathbf{v} \in H^1(\Omega)^2. \tag{3.14}$$

Theorem 3.3 (First Korn Inequality) *There exists a positive constant C such that*

$$\|\mathbf{v}\|_{1,\Omega} \le C\, \|\boldsymbol{\epsilon}(\mathbf{v})\|_{0,\Omega} \quad \forall\, \mathbf{v} \in H^1_{0,D}(\Omega)^2. \tag{3.15}$$

In order to prove the validity of Korn's inequalities, we need the following useful lemma (see [2]).

Lemma 3.1 *There exists a positive constant C such that for all $q \in L^2(\Omega)$, there is a $\mathbf{v} \in H^1(\Omega)^2$ satisfying*

$$\nabla \cdot \mathbf{v} = q \text{ in } \Omega \quad \text{and} \quad \|\mathbf{v}\|_{1,\Omega} \le C\, \|q\|_{0,\Omega}.$$

If furthermore, $q \in L^2_0(\Omega)$, we may assume that $\mathbf{v} \in H^1_0(\Omega)$.

Note that the skew-symmetric part of gradient of vector field may be represented by the curl of the vector field

$$\frac{1}{2}\left(\nabla \mathbf{v} - (\nabla \mathbf{v})^t\right) = \frac{1}{2}(\nabla\times \mathbf{v})\, \boldsymbol{\chi} \quad \text{with} \quad \boldsymbol{\chi} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence,

$$\nabla \mathbf{v} = \boldsymbol{\epsilon}(\mathbf{v}) + \frac{1}{2}\left(\nabla \mathbf{v} - (\nabla \mathbf{v})^t\right) = \boldsymbol{\epsilon}(\mathbf{v}) + \frac{1}{2}(\nabla\times \mathbf{v})\,\boldsymbol{\chi}. \tag{3.16}$$

Proof of Second Korn Inequality: For any $\mathbf{v} \in \hat{H}^1(\Omega)^2$, it follows from the fact that $(\nabla \mathbf{v},\, \nabla^\perp \mathbf{w}) = 0$ for any $\mathbf{w} \in H_0^1(\Omega)^2$ and (3.16) that

$$\begin{aligned}
\|\nabla \mathbf{v}\|_{0,\Omega}^2 &= (\nabla \mathbf{v},\, \nabla \mathbf{v}) = (\nabla \mathbf{v},\, \nabla \mathbf{v} - \nabla^\perp \mathbf{w}) \\
&= (\boldsymbol{\epsilon}(\mathbf{v}),\, \nabla \mathbf{v} - \nabla^\perp \mathbf{w}) + \frac{1}{2}\big((\nabla\times \mathbf{v})\,\boldsymbol{\chi},\, \nabla \mathbf{v} - \nabla^\perp \mathbf{w}\big) \\
&= (\boldsymbol{\epsilon}(\mathbf{v}),\, \nabla \mathbf{v} - \nabla^\perp \mathbf{w}) + \frac{1}{2}\big(\nabla\times \mathbf{v},\, \nabla\times \mathbf{v} - \nabla\cdot\, \mathbf{w}\big).
\end{aligned}$$

Since $\nabla\times \mathbf{v} \in L_0^2(\Omega)$, by Lemma 3.1 one can choose $\mathbf{w} \in H_0^1(\Omega)^2$ such that

$$\nabla\cdot \mathbf{w} = \nabla\times\mathbf{v} \text{ in } \Omega \quad \text{and} \quad \|\mathbf{w}\|_{1,\Omega} \le C\,\|\nabla\times\mathbf{v}\|_{0,\Omega} \le C\,\|\mathbf{v}\|_{1,\Omega}.$$

Therefore, by the Cauchy-Schwarz, triangle, and Poincaré inequalities we have that

$$\begin{aligned}
\|\nabla \mathbf{v}\|_{0,\Omega}^2 &= (\boldsymbol{\epsilon}(\mathbf{v}),\, \nabla \mathbf{v} - \nabla^\perp \mathbf{w}) \le \|\boldsymbol{\epsilon}(\mathbf{v})\|_{0,\Omega}\,\|\nabla \mathbf{v} - \nabla^\perp \mathbf{w}\|_{0,\Omega} \\
&\le C\,\|\boldsymbol{\epsilon}(\mathbf{v})\|_{0,\Omega}\,\|\mathbf{v}\|_{1,\Omega} \le C\,\|\boldsymbol{\epsilon}(\mathbf{v})\|_{0,\Omega}\,\|\nabla \mathbf{v}\|_{0,\Omega}.
\end{aligned}$$

Dividing by $\|\nabla \mathbf{v}\|_{0,\Omega}$ on both sides of the above inequality yields the second Korn inequality in (3.13). This completes the proof of Theorem 3.1.

Homework: show that $(\nabla \mathbf{v},\, \nabla^\perp \mathbf{w}) = 0$ for any $\mathbf{w} \in H_0^1(\Omega)^2$ and that $\boldsymbol{\chi} : \nabla\mathbf{v} = \nabla\times \mathbf{v}$ and $\boldsymbol{\chi} : \nabla^\perp\mathbf{w} = \nabla\cdot\, \mathbf{w}$.

Proof of Korn's Inequality: For any $\mathbf{v} \in H^1(\Omega)^2$, we have

$$\mathbf{v} = \mathbf{z} + \mathbf{w} \quad \text{with} \quad \mathbf{z} \in \hat{H}^1(\Omega)^2,\ \mathbf{w} \in \mathrm{RM}.$$

The second Korn inequality gives

$$\|\mathbf{z}\|_{1,\Omega} \le C\,\|\boldsymbol{\epsilon}(\mathbf{z})\|_{0,\Omega} = C\,\|\boldsymbol{\epsilon}(\mathbf{v})\|_{0,\Omega}. \tag{3.17}$$

Since RM is a finite dimensional space, then it follows from the fact that all norms defined on RM are equivalent, the triangle inequality, and (3.17) that

$$\|\mathbf{w}\|_{1,\Omega} \le C\,\|\mathbf{w}\|_{0,\Omega} \le C\,\left(\|\mathbf{v}\|_{0,\Omega} + \|\mathbf{z}\|_{0,\Omega}\right) \le C\,\left(\|\mathbf{v}\|_{0,\Omega} + \|\boldsymbol{\epsilon}(\mathbf{v})\|_{0,\Omega}\right).$$

Now, (3.14) is then a consequence of the triangle inequality and above two inequalities.

Another Proof of Korn's Inequality: We shall prove the validity of Korn's inequality (3.14) by contradictory argument. To this end, assume that (3.14) does not hold. This implies that there exists a sequence $\{\mathbf{v}_n\} \subset H^1(\Omega)^2$ such that

$$\|\mathbf{v}_n\|_{1,\Omega} = 1 \quad \text{and} \quad \|\mathbf{v}_n\|_{0,\Omega} + \|\boldsymbol{\epsilon}(\mathbf{v}_n)\|_{0,\Omega} < \frac{1}{n}. \tag{3.18}$$

For each n, by (3.8) and (3.10) let

$$\mathbf{v}_n = \mathbf{z}_n + \mathbf{w}_n \quad \text{with} \quad \|\mathbf{z}_n\|_{1,\Omega} + \|\mathbf{w}_n\|_{1,\Omega} \le C\,\|\mathbf{v}_n\|_{1,\Omega} = C, \tag{3.19}$$

where $\mathbf{z}_n \in \hat{H}^1(\Omega)^2$ and $\mathbf{w}_n \in \mathrm{RM}$. Since $\boldsymbol{\epsilon}(\mathbf{w}_n) = \mathbf{0}$, using the second Korn inequality in (3.13) and (3.19) we have

$$C\,\|\mathbf{z}_n\|_{1,\Omega} \le \|\boldsymbol{\epsilon}(\mathbf{z}_n)\|_{0,\Omega} = \|\boldsymbol{\epsilon}(\mathbf{v}_n) - \boldsymbol{\epsilon}(\mathbf{w}_n)\|_{0,\Omega} = \|\boldsymbol{\epsilon}(\mathbf{v}_n)\|_{0,\Omega} < \frac{1}{n}.$$

Hence,

$$\lim_{n\to\infty} \|\mathbf{z}_n\|_{1,\Omega} = 0. \tag{3.20}$$

It follows from (3.19) that $\{\mathbf{w}_n\} \subset \mathrm{RM}$ is a bounded sequence in $H^1(\Omega)^2$. Since RM is a three dimensional space, $\{\mathbf{w}_n\}$ has a convergent subsequence $\{\mathbf{w}_{n_j}\}$ in $H^1(\Omega)^2$: there exists $\mathbf{w} \in \mathrm{RM}$ such that

$$\lim_{j\to\infty} \mathbf{w}_{n_j} = \mathbf{w} \quad \text{in } H^1(\Omega)^2.$$

By (3.20), this gives

$$\lim_{j\to\infty} \mathbf{v}_{n_j} = \lim_{j\to\infty} \left(\mathbf{z}_{n_j} + \mathbf{w}_{n_j}\right) = \lim_{j\to\infty} \mathbf{w}_{n_j} = \mathbf{w} \quad \text{in } H^1(\Omega)^2.$$

Now, by (3.18), $\|\mathbf{v}_{n_j}\|_{1,\Omega} = 1$ implies

$$\|\mathbf{w}\|_{1,\Omega} = 1,$$

and $\|\mathbf{v}_{n_j}\|_{0,\Omega} < 1/n_j$ implies

$$\|\mathbf{w}\|_{0,\Omega} = 0.$$

This is contradictory and, hence, completes the proof of Korn's inequality (3.14).

Proof of First Korn Inequality: The same proof by contradiction yields that there exists $\mathbf{w} \in \mathrm{RM} \cap H^1_{0,D}(\Omega)^2$ such that $\|\mathbf{w}\|_{1,\Omega} = 1$. But the facts that $\mathbf{w} \in \mathrm{RM}$ and that $\mathbf{w} = \mathbf{0}$ on Γ_D imply that $\mathbf{w} = \mathbf{0}$ in Ω. This is contradictory and, hence, completes the proof of the first Korn inequality in (3.15).

Homework: furnish details of this proof.

Theorem 3.4 *The variational problem in (3.7) has a unique solution* $\mathbf{u} \in H^1_{0,D}(\Omega)^2$. *The variational problem in (3.11) has a unique solution* $\mathbf{u} \in \hat{H}^1(\Omega)^2$. *For both problems, the solution satisfies the following a priori estimate:*

$$\|\mathbf{u}\|_{1,\Omega} + \lambda^{\frac{1}{2}} \|\nabla \cdot \mathbf{u}\|_{0,\Omega} \le C \left(\|\mathbf{f}\|_{-1,D} + \|\mathbf{g}\|_{-1/2,\Gamma_N} \right). \tag{3.21}$$

If $\Gamma_D = \partial\Omega$ *or* $\Gamma_N = \partial\Omega$, *then the above estimate may be improved as follows:*

$$\|\mathbf{u}\|_{1,\Omega} + \lambda \|\nabla \cdot \mathbf{u}\|_{0,\Omega} \le C \left(\|\mathbf{f}\|_{-1,D} + \|\mathbf{g}\|_{-1/2,\Gamma_N} \right); \tag{3.22}$$

and, moreover, when the boundary is either sufficiently smooth or convex polygon, then the solution $\mathbf{u}$ *is in* $H^2(\Omega)^2$ *satisfying*

$$\|\mathbf{u}\|_{2,\Omega} + \lambda \|\nabla \cdot \mathbf{u}\|_{1,\Omega} \le C \left(\|\mathbf{f}\|_{0,\Omega} + \|\mathbf{g}\|_{1/2,\Gamma_N} \right). \tag{3.23}$$

Proof: It is easy to show that the linear form is continuous

$$\begin{aligned} |f(\mathbf{v})| &\le \left(\|\mathbf{f}\|_{-1,\Omega} + \|\mathbf{g}\|_{-1/2,\Gamma_N} \right) \|\mathbf{v}\|_{1,\Omega} \quad \forall\, \mathbf{v} \in H^1_{0,D}(\Omega)^2 \\ |f(\mathbf{v})| &\le \left(\|\mathbf{f}\|_{-1,\Omega} + \|\mathbf{g}\|_{-1/2,\partial\Omega} \right) \|\mathbf{v}\|_{1,\Omega} \quad \forall\, \mathbf{v} \in \hat{H}^1(\Omega)^2 \end{aligned}$$

and that the bilinear form is continuous

$$|a(\mathbf{u}, \mathbf{v})| \le C\,(\lambda + 1)\, \|\mathbf{v}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \quad \forall\, \mathbf{u}, \mathbf{v} \in H^1_{0,D}(\Omega)^2 \text{ or } \forall\, \mathbf{u}, \mathbf{v} \in \hat{H}^1(\Omega)^2.$$

It follows from (3.15) and (3.13) that the bilinear form is coercive:

$$a(\mathbf{v}, \mathbf{v}) \ge \alpha \left(\|\mathbf{v}\|^2_{1,\Omega} + \lambda \|\nabla \cdot \mathbf{v}\|^2_{0,\Omega} \right) \ge \alpha \|\mathbf{v}\|^2_{1,\Omega} \tag{3.24}$$

for any $\mathbf{v} \in H^1_{0,D}(\Omega)^2$ or any $\mathbf{v} \in \hat{H}^1(\Omega)^2$. Now, the Lax-Milgram theorem implies that both problems in (3.7) and (3.11) are well-posed.

To prove the *a priori* estimate in (3.21), choosing $\mathbf{v} = \mathbf{u}$ in (3.7) gives

$$\begin{aligned} a(\mathbf{u}, \mathbf{u}) &= f(\mathbf{u}) \le \|\mathbf{f}\|_{-1,D} \|\mathbf{u}\|_{1,\Omega} + \|\mathbf{g}\|_{-1/2,\Gamma_N} \|\mathbf{u}\|_{1/2,\Gamma_N} \\ &\le \left(\|\mathbf{f}\|_{-1,D} + \|\mathbf{g}\|_{-1/2,\Gamma_N} \right) \|\mathbf{v}\|_{1,\Omega}. \end{aligned}$$

Now, (3.21) follows from (3.24).

We only prove (3.22) for $\Gamma_D = \partial\Omega$ since the proof is similar when $\Gamma_N = \partial\Omega$. By Lemma 3.1 and the fact that $\nabla \cdot \mathbf{u} \in L^2_0(\Omega)$, there exists a $\mathbf{w} \in H^1_0(\Omega)^2$ such that

$$\nabla \cdot \mathbf{w} = \nabla \cdot \mathbf{u} \text{ in } \Omega \quad \text{and} \quad \|\mathbf{w}\|_{1,\Omega} \le C \|\nabla \cdot \mathbf{u}\|_{0,\Omega}.$$

Now, choosing $\mathbf{v} = \mathbf{w}$ in (3.7), using the Cauchy-Schwarz inequality, and (3.21) give

$$\begin{aligned} \lambda \|\nabla \cdot \mathbf{u}\|^2_{0,\Omega} &= f(\mathbf{w}) - 2\mu\,(\boldsymbol{\epsilon}(\mathbf{u}), \boldsymbol{\epsilon}(\mathbf{w})) \\ &\le \|\mathbf{f}\|_{-1,0} \|\mathbf{w}\|_{1,\Omega} + 2\mu \|\boldsymbol{\epsilon}(\mathbf{u})\|_{0,\Omega} \|\boldsymbol{\epsilon}(\mathbf{w})\|_{0,\Omega} \\ &\le C \|\mathbf{f}\|_{-1,0} \|\nabla \cdot \mathbf{u}\|_{0,\Omega}. \end{aligned}$$

Hence,

$$\lambda \|\nabla \cdot \mathbf{u}\|_{0,\Omega} \le C \|\mathbf{f}\|_{-1,0},$$

which, together with (3.21), implies (3.22). For the proof of (3.23), see [1]. ■

4 Finite Element Approximation Based on Displacement Model

Let $V_h \subset V$ be a finite element space satisfying

$$\inf_{\boldsymbol{\phi} \in V_h} \|\mathbf{v} - \boldsymbol{\phi}\|_{1,\Omega} \le C\, h^s\, |\mathbf{v}|_{s+1,\Omega},$$

where V is either $H^1_{0,D}(\Omega)^d$ for $\mathrm{mes}(\Gamma_{0,D}) \neq 0$ or $\hat{H}^1(\Omega)^d$ otherwise. Finite element approximation is to find $\mathbf{u} \in V_h$ such that

$$a(\mathbf{u}_h,\, \mathbf{v}) = f(\mathbf{v}) \quad \forall\, \mathbf{v} \in V_h \tag{4.25}$$

It follows from Cea's Lemma and the approximation property that

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \le C\,(1+\lambda) \inf_{\boldsymbol{\phi} \in V_h} \|\mathbf{u} - \boldsymbol{\phi}\|_{1,\Omega} \le C\,(1+\lambda)\, h^s\, |\mathbf{v}|_{s+1,\Omega}.$$

Note that this error bound depends on λ!!! Non-confirming finite element, the reduce integration, or the perturbed Stokes equations lead to finite element approximations being uniform in λ. These three approaches are equivalent in some sense. See [2] for some references.

References

[1] S. C. Brenner and L. Sung, *Linear finite elements for planar linear elasticity*, Math. Comp., 59 (1992), 321-338.

[2] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, New York, 1994.

Homework

1. Define curl operator in four dimensions.
2. Compute $\mathcal{A}_\lambda = \mathcal{C}_\lambda^{-1}$.
3. Prove the Lax-Milgram theorem.
4. Define the space of infinitesimal rigid motions in three dimensions.
5. Derive (3.9) and prove (3.10).
6. Show that $(\nabla\, \mathbf{v},\, \nabla^\perp\, \mathbf{w}) = 0$ for any $\mathbf{w} \in H^1_0(\Omega)^2$.
7. Show that $\boldsymbol{\chi} : \nabla \mathbf{v} = \nabla \times \mathbf{v}$ and that $\boldsymbol{\chi} : \nabla^\perp \mathbf{w} = \nabla \cdot\, \mathbf{w}$.
8. Furnish details of proof of the first Korn inequality.

Lecture Note III: Least-Squares Method

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In this chapter, we shall present least-squares methods for second-order scalar partial differential equations, elastic equations of solids, incompressible Newtonian fluid flow, and Maxwell's equations in electromagnetic.

1 A General Methodology

We give a general methodology for the design of least-squares methods applied to a first-order system of partial differential equations. Consider the following first-order partial differential system defined on a bounded domain $\Omega \subset R^d$ ($d = 2$ or 3):

$$\left\{ \begin{array}{rcll} \mathcal{L}\mathcal{U} & = & \mathcal{F} & \text{in } \Omega, \\ \mathcal{B}\mathcal{U} & = & \mathcal{G} & \text{on } \partial\Omega, \end{array} \right. \tag{1.1}$$

where $\mathcal{L} = (L_{ij})_{m\times n}$ is a block $m \times n$ matrix differential operator of at most first order, $\mathcal{B} = (B_{ij})_{l\times n}$ is a block $l \times n$ matrix operator, $\mathcal{U} = (U_i)_{n\times 1}$ is unknown, $\mathcal{F} = (F_i)_{m\times 1}$ is a given block vector-valued function defined in Ω, $\mathcal{G} = (G_i)_{l\times 1}$ is a given block vector-valued function defined on $\partial\Omega$. Assume that first-order system (1.1) has a unique solution $\mathcal{U}$. Boundary conditions in a least-squares formulation can be imposed either strongly (in the solution space) or weakly (by adding boundary functionals). For simplicity of presentation, we impose them in the solution space Φ. Assume that Φ is appropriately chosen so that least-squares functional is well defined.

Define the least-squares functional by

$$G(\mathcal{U}\,;\,\mathcal{F}) = \sum_{i=1}^{m} \| \sum_{j=1}^{n} L_{ij}\,U_j - F_i\|^2_{k(i),\Omega}, \tag{1.2}$$

where $\|\cdot\|_{k(i),\Omega}$ denotes a Sobolev norm and $k(i) = -1$ or 0. If $k(i) = 0$ for all i, $G(\mathcal{U}\,;\,\mathcal{F})$ is referred to as *the L^2 norm least-squares functional*, otherwise, it is referred to as *the inverse norm least-squares functional*. Denote the Laplace operator by Δ and the L^2

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inner product by $(f,\,g)=\int_\Omega f\,g\,dx$. Then $\|\cdot\|_{L^2(\Omega)}=\|\cdot\|_{H^0(\Omega)}=\sqrt{(\cdot,\,\cdot)}$ and $\|\cdot\|_{H^{-1}(\Omega)}=\sqrt{((-\Delta)^{-1}\cdot,\,\cdot)}$. Now, the least-squares minimization problem is to minimize the least-squares functional over Φ:

$$G(\mathcal{U}\,;\,\mathcal{F})=\min_{\mathcal{V}\in\Phi} G(\mathcal{V}\,;\,\mathcal{F}). \tag{1.3}$$

This is equivalent to solving the normal equation:

$$(\mathcal{L}^*\,\mathcal{K}\,\mathcal{L})\,\mathcal{U}=(\mathcal{L}^*\,\mathcal{K})\,\mathcal{F} \tag{1.4}$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$ with respect to the L^2 inner product and $\mathcal{K}$ is a block diagonal operator with each block associated with the $H^{k(i)}(\Omega)$ norm

$$\mathcal{K}=\text{diag}\,(K_1,\,\cdots,\,K_m)\quad\text{where}\quad K_i=I\ \text{ or }(-\Delta)^{-1}.$$

That is, each diagonal block of $\mathcal{K}$ is either the identity or the inverse of Laplacian. (For the L^2 least squares, we have $\mathcal{K}=I$.) The normal operator $\mathcal{L}^*\,\mathcal{K}\,\mathcal{L}$ is a differential operator of at most second-order. The variational form of (1.4) is to find $\mathcal{U}\in\Phi$ such that

$$b(\mathcal{U},\,\mathcal{V})\equiv(\mathcal{K}\,\mathcal{L}\mathcal{U},\,\mathcal{L}\,\mathcal{V})=(\mathcal{K}\,\mathcal{F},\,\mathcal{L}\,\mathcal{V})\equiv f(\mathcal{V})\quad\forall\,\mathcal{V}\in\Phi. \tag{1.5}$$

It is easy to check that

$$G(\mathcal{V}\,;\,\mathbf{0})=b(\mathcal{V},\,\mathcal{V}).$$

The design of the least-squares method is to choose *first-order system* (1.1) and *least-squares norms* so that the least-squares problem (i.e., the normal equation or the minimization problem or the weak form) can be numerically solved *effectively* and *efficiently*; i.e.,

- the least-squares variables can be discretized with *optimal accuracy*;
- the resulting algebraic system can be solved with *optimal complexity*.

It is well known that problems with the identity or Laplace operators can be numerically solved with both optimal accuracy and complexity. Recently, it was shown that this is also true for problems involving the $H(\text{div})$ and $H(\text{curl})$ operators

$$H_{\text{div}}=I-\nabla\,\text{div}\quad\text{and}\quad H_{\text{curl}}=I+\nabla\times\nabla\times$$

provided Raviart-Thomas elements [17] are used for $H(\text{div})$ and edge elements [3] are used for $H(\text{curl})$. Hence, one wants to develop the least-squares method so that the normal operator $\mathcal{L}^*\,\mathcal{K}\,\mathcal{L}$ is equivalent to a block diagonal operator whose diagonal block is either the identity, Laplacian, H_{div}, or H_{curl} operators:

$$\mathcal{D}=\text{diag}\,(D_1,\,\cdots,\,D_m)\quad\text{where}\quad D_i=I,\,-\Delta,\,H_{\text{div}},\ \text{ or }H_{\text{curl}}.$$

With such equivalence, the least-squares problem can then be numerically solved with optimal accuracy and optimal complexity. Moreover, finite element spaces for different variables U_i can be chosen *independently* and, hence, based solely on the approximation properties and implementation/computational costs.

The above equivalence between $\mathcal{L}^* \mathcal{K} \mathcal{L}$ and $\mathcal{D}$ means that there exist positive constants α_0 and α_1 such that

$$\alpha_0 \sum_{i=1}^m \|V_i\|_{D_i}^2 \le b(\mathcal{V}, \mathcal{V}) \quad \text{and} \quad b(\mathcal{U}, \mathcal{V}) \le \alpha_1 \left(\sum_{i=1}^m \|U_i\|_{D_i}^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^m \|V_i\|_{D_i}^2\right)^{\frac{1}{2}} \tag{1.6}$$

for all $\mathcal{U}, \mathcal{V} \in \Phi$.

Here $\|\cdot\|_{D_i}$ denotes the L^2, H^1, $H(\mathrm{div})$, or $H(\mathrm{curl})$ norms. The main task of analyzing least-squares methods is to establish (1.6). Many physical models involve parameters such as the Lamé constants for solids and the viscosity parameters for fluids. It is then important to establish equivalence independent of these parameters. This is because parameter-independent equivalence implies *robustness* of the least-squares methods with respect to these parameters.

1.1 Least-Squares Approximation

Assume that Φ^h is a finite dimensional subspace of Φ satisfying the following approximation property:

$$\inf_{\mathcal{V}^h \in \Phi^h} \left(\sum_{i=1}^m \|V_i - V_i^h\|_{D_i}^2\right)^{\frac{1}{2}} \le C_a\, h^s \tag{1.7}$$

for all $\mathcal{V} = (V_i)_{n\times 1} \in \Phi$. Then least-squares approximation is to find $\mathcal{U}^h \in \Phi^h$ such that

$$G(\mathcal{U}^h\,;\,\mathcal{F}) = \min_{\mathcal{V}\in\Phi^h} G(\mathcal{V}\,;\,\mathcal{F}). \tag{1.8}$$

Equivalently, find $\mathcal{U}^h \in \Phi^h$ such that

$$b(\mathcal{U}^h, \mathcal{V}) = f(\mathcal{V}) \quad \forall\, \mathcal{V} \in \Phi^h. \tag{1.9}$$

Theorem 1.1 *Let $\mathcal{U}$ and $\mathcal{U}^h$ be the solutions of* (1.5) *and* (1.9)*, respectively. Assume that equivalence* (1.6) *and approximation property* (1.7) *hold. Then we have the following error estimation:*

$$\left(\sum_{i=1}^m \|U_i - U_i^h\|_{D_i}^2\right)^{\frac{1}{2}} \le C_a \frac{\alpha_1}{\alpha_0}\, h^s. \tag{1.10}$$

Proof: Difference of (1.5) and (1.9) gives the error equation:

$$b(\mathcal{U} - \mathcal{U}^h, \mathcal{V}) = 0 \quad \forall\, \mathcal{V} \in \Phi^h. \tag{1.11}$$

It follows from (1.6) and (1.7) that for any $\mathcal{V} \in \Phi^h$

$$\begin{aligned}
\alpha_0 \sum_{i=1}^{m} \|U_i - U_i^h\|_{D_i}^2 &\le b(\mathcal{U} - \mathcal{U}^h,\, \mathcal{U} - \mathcal{U}^h) = b(\mathcal{U} - \mathcal{U}^h,\, \mathcal{U} - \mathcal{V}) \\
&\le \alpha_1 \left(\sum_{i=1}^{m} \|U_i - U_i^h\|_{D_i}^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} \|U_i - V_i\|_{D_i}^2 \right)^{\frac{1}{2}} \\
&\le \alpha_1\, C_a\, h^s \left(\sum_{i=1}^{m} \|U_i - U_i^h\|_{D_i}^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Dividing on both sides by $\alpha_0 \left(\sum_{i=1}^{m} \|U_i - U_i^h\|_{D_i}^2\right)^{\frac{1}{2}}$ yields (1.10) and, hence, the theorem. ■

1.2 Mesh Refinement Indicator

Let $\mathcal{U}$ be the solution of (1.1) and $\mathcal{V} \in \Phi$ be a computed approximation to $\mathcal{U}$. Then (1.1) and (1.6) imply

$$\begin{aligned}
G(\mathcal{V};\, \mathcal{F}) &= \sum_{i=1}^{m} \| \sum_{j=1}^{n} L_{ij}\, V_j - F_i \|_{k(i),\Omega}^2 = \sum_{i=1}^{m} \| \sum_{j=1}^{n} L_{ij}\, (V_j - U_j) \|_{k(i),\Omega}^2 \\
&\sim \sum_{i=1}^{m} \|V_i - U_i\|_{D_i}^2.
\end{aligned} \tag{1.12}$$

Since

$$G(\mathbf{0};\, \mathcal{F}) = \sum_{i=1}^{m} \|F_i\|_{k(i),\Omega}^2 = \sum_{i=1}^{m} \| \sum_{j=1}^{n} L_{ij}\, U_j \|_{k(i),\Omega}^2 \sim \sum_{i=1}^{m} \|U_i\|_{D_i}^2,$$

combining with (1.12) gives

$$\frac{G(\mathcal{V};\, \mathcal{F})}{G(\mathbf{0};\, \mathcal{F})} \sim \frac{\sum_{i=1}^{m} \|V_i - U_i\|_{D_i}^2}{\sum_{i=1}^{m} \|U_i\|_{D_i}^2}. \tag{1.13}$$

(1.12) means that the value of the least-squares functional at $\mathcal{V}$ gives certain measurement of absolute difference between the solution $\mathcal{U}$ and an approximation $\mathcal{V}$ in the functional induced norm. Therefore, the value of the least-squares functional at $\mathcal{V}$ on each element probably gives a reasonable mesh refinement indicator. Especially, this is true for nonlinear problem.

2 Second-Order Scalar PDEs

Consider the following second-order elliptic boundary value problem:

$$\left\{ \begin{array}{rcll} -\nabla \cdot \, (A \nabla \, p) + X p & = & f, & \text{in} \quad \Omega, \\ p & = & 0, & \text{on} \quad \Gamma_D, \\ \mathbf{n} \cdot A \nabla \, p & = & 0, & \text{on} \quad \Gamma_N, \end{array} \right. \tag{2.1}$$

where A is a $d \times d$ symmetric matrix of functions in $L^2(\Omega)$ and X is an at most first-order linear differential operator. We assume that A is uniformly symmetric positive definite: there exist positive constants $0 < \lambda \leq \Lambda$ such that

$$\lambda \boldsymbol{\xi}^T \boldsymbol{\xi} \leq \boldsymbol{\xi}^T A \boldsymbol{\xi} \leq \Lambda \boldsymbol{\xi}^T \boldsymbol{\xi} \tag{2.2}$$

for all $\boldsymbol{\xi} \in \Re^d$ and almost all $x \in \overline{\Omega}$. The corresponding variational form of system (2.1) is to find $p \in V$ such that

$$a(p,\, q) = f(q) \quad \forall \; q \in V \tag{2.3}$$

where

$$V = \left\{ \begin{array}{ll} H^1_{0,D}(\Omega) & \text{if } \operatorname{mes}(\Gamma_D) \neq \emptyset \\ \hat{H}^1(\Omega) & \text{otherwise,} \end{array} \right.$$

with $\hat{H}^1(\Omega) = \{v \in H^1(\Omega) \,|\, \int_\Omega v\, dx = 0\}$ and the bilinear and linear forms are defined by

$$a(p,\, q) = (A\nabla p,\, \nabla q) + (Xp,\, q) \quad \text{and} \quad f(q) = (f,\, q),$$

respectively. Under appropriate assumptions on Γ_D and X, problem (2.3) is uniquely solvable in $H^1_{0,D}(\Omega)$ for any $f \in H^{-1}(\Omega)$ or uniquely solvable in $\hat{H}^1(\Omega)$ if and only if f satisfies the compatibility condition

$$\int_\Omega f\, dx = 0.$$

2.1 First-Order System of PDEs

For (2.1), we consider two first-order systems. To this end, introducing the flux variable

$$\mathbf{u} = -A\nabla p,$$

problem (2.1) may be rewritten as a first-order system of partial differential equations as follows:

$$\mathcal{L}\mathcal{U} \equiv \begin{pmatrix} A^{-\frac{1}{2}} & A^{\frac{1}{2}}\nabla \\ \operatorname{div} & X \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ f \end{pmatrix} \equiv \mathcal{F} \quad \text{in } \Omega \tag{2.4}$$

with boundary conditions

$$p = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n} \cdot \mathbf{u} = 0 \quad \text{on } \Gamma_N. \tag{2.5}$$

Based on this system, we will consider two functionals: (1) Div least-squares functional and (2) inverse norm least-squares functional.

Note that if $\mathbf{u}$ is sufficiently smooth, then the properly scaled solution, $A^{-1}\mathbf{u}$, of (2.4) is curl free, i.e., $\nabla\times(A^{-1}\mathbf{u}) = \mathbf{0}$, and that the homogeneous Dirichlet boundary condition on Γ_D implies the tangential flux condition

$$\mathbf{n} \times (A^{-1}\mathbf{u}) = \mathbf{0} \quad \text{on } \Gamma_D.$$

We then have a redundant but consistent first-order system:

$$\mathcal{L}\mathcal{U} \equiv \begin{pmatrix} A^{-\frac{1}{2}} & A^{\frac{1}{2}}\nabla \\ \text{div} & X \\ \nabla\times\ A^{-1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ f \\ \mathbf{0} \end{pmatrix} \equiv \mathcal{F} \tag{2.6}$$

with boundary conditions

$$p = 0, \quad \mathbf{n} \times (A^{-1}\mathbf{u}) = \mathbf{0} \quad \text{on } \Gamma_D, \quad \text{and} \quad \mathbf{n}\cdot\mathbf{u} = 0 \quad \text{on } \Gamma_N. \tag{2.7}$$

Based on this system, we will consider Div-curl least-squares functional.

2.2 Div Least-Squares Functional

Let $H_N(\text{div};\,\Omega)$ denote a subspace of $H(\text{div};\,\Omega)$:

$$H_N(\text{div};\,\Omega) = \{\mathbf{v} \in H(\text{div};\,\Omega) \,:\, \mathbf{n}\cdot\mathbf{v} = 0 \text{ on } \Gamma_N\}.$$

For any $(\mathbf{v},\,q) \in H_N(\text{div};\,\Omega) \times V \equiv \Phi$, consider the following div least-squares functional:

$$G(\mathbf{v},\,q;\,f) = \|A^{-\frac{1}{2}}(\mathbf{v} + A\nabla q)\|^2_{0,\Omega} + \|\nabla\cdot\mathbf{v} + Xq - f\|^2_{0,\,\Omega}. \tag{2.8}$$

The corresponding normal operator is

$$\mathcal{L}^*\,\mathcal{L} = \begin{pmatrix} A^{-1} - \nabla\,\text{div} & \nabla\,(I - X) \\ -(I - X^*)\,\text{div} & -\text{div}\,A\,\nabla + X^*\,X \end{pmatrix} \tag{2.9}$$

with $\mathcal{L}$ defined in (2.4) and the corresponding bilinear and linear forms are

$$b(\mathbf{u},\,p;\,\mathbf{v},\,q) \;=\; (A^{-1}(\mathbf{u} + A\nabla p),\,(\mathbf{v} + A\nabla q)) + (\nabla\cdot\mathbf{u} + Xp,\,\nabla\cdot\mathbf{v} + Xq) \tag{2.10}$$

$$f(\mathbf{v},\,q) \;=\; (f,\,\nabla\cdot\mathbf{v} + Xq). \tag{2.11}$$

The main task of this section is to establish the following equivalence:

$$\mathcal{L}^*\,\mathcal{L} \sim \begin{pmatrix} I - \nabla\,\text{div} & \mathbf{0} \\ \mathbf{0} & -\Delta \end{pmatrix}.$$

Theorem 2.1 *There exist positive constants* α_0 *and* α_1 *such that*

$$\alpha_0 \left(\|\mathbf{v}\|^2_{H(\mathrm{div})} + \|q\|^2_{1,\Omega} \right) \le b(\mathbf{v},\, q;\, \mathbf{v},\, q) = G(\mathbf{v},\, q;\, 0) \tag{2.12}$$

for any $(\mathbf{v},\, q) \in H_N(\mathrm{div};\,\Omega) \times V$ *and*

$$b(\mathbf{u},\, p;\, \mathbf{v},\, q) \le \alpha_1 \left(\|\mathbf{u}\|^2_{H(\mathrm{div})} + \|p\|^2_{1,\Omega} \right)^{\frac{1}{2}} \left(\|\mathbf{v}\|^2_{H(\mathrm{div})} + \|q\|^2_{1,\Omega} \right)^{\frac{1}{2}} \tag{2.13}$$

for any $(\mathbf{u},\, p)$ *and any* $(\mathbf{v},\, q)$ *in* $H_N(\mathrm{div};\,\Omega) \times V$.

Proof: (2.13) is a direct consequence of the Cauchy-Schwarz and triangle inequalities. To show the validity of (2.12), we first establish that

$$\left(\|\mathbf{v}\|^2_{H(\mathrm{div})} + \|q\|^2_{1,\Omega} \right) \le C \left(G(\mathbf{v},\, q;\, 0) + \|q\|^2_{0,\Omega} \right). \tag{2.14}$$

It follows from integration by parts, the Cauchy-Schwarz inequality, the Poincaré inequality, and (2.2) that

$$\begin{aligned}
&\|A^{\frac{1}{2}} \nabla q\|^2_{0,\Omega} = (A^{\frac{1}{2}} \nabla q + A^{-\frac{1}{2}} \mathbf{v},\, A^{\frac{1}{2}} \nabla q) - (\mathbf{v},\, \nabla q) \\
= \;& (A^{\frac{1}{2}} \nabla q + A^{-\frac{1}{2}} \mathbf{v},\, A^{\frac{1}{2}} \nabla q) + (\nabla \cdot \mathbf{v},\, q) \\
= \;& (A^{\frac{1}{2}} \nabla q + A^{-\frac{1}{2}} \mathbf{v},\, A^{\frac{1}{2}} \nabla q) + (\nabla \cdot \mathbf{v} + Xq,\, q) - (Xq,\, q) \\
\le \;& \|A^{\frac{1}{2}} \nabla q + A^{-\frac{1}{2}} \mathbf{v}\|_{0,\Omega} \, \|A^{\frac{1}{2}} \nabla q\|_{0,\Omega} + \|\nabla \cdot \mathbf{v} + Xq\|_{0,\Omega} \, \|q\|_{0,\Omega} + \|Xq\|_{0,\Omega} \, \|q\|_{0,\Omega} \\
\le \;& \left(\|A^{\frac{1}{2}} \nabla q + A^{-\frac{1}{2}} \mathbf{v}\|_{0,\Omega} + C \, \|q\|_{0,\Omega} \right) \|A^{\frac{1}{2}} \nabla q\|_{0,\Omega} + \|\nabla \cdot \mathbf{v} + Xq\|_{0,\Omega} \|q\|_{0,\Omega}.
\end{aligned}$$

Combining the fact that $ab \le \frac{1}{2}\, a^2 + \frac{1}{2}\, b^2$, we have

$$\|q\|^2_{1,\Omega} \le C \, \|A^{\frac{1}{2}} \nabla q\|^2_{0,\Omega} \le C \left(G(\mathbf{v},\, q;\, 0) + \|q\|^2_{0,\Omega} \right). \tag{2.15}$$

(2.2), the triangle inequality, and (2.15) give

$$\begin{aligned}
\|\mathbf{v}\|^2_{0,\Omega} \;&\le\; \frac{1}{\Lambda} \, \|A^{-\frac{1}{2}} \mathbf{v}\|^2_{0,\Omega} \le \frac{2}{\Lambda} \left(\|A^{-\frac{1}{2}} \mathbf{v} + A^{\frac{1}{2}} \nabla q\|^2_{0,\Omega} + \|A^{\frac{1}{2}} \nabla q\|^2_{0,\Omega} \right) \\
&\le\; C \left(G(\mathbf{v},\, q;\, 0) + \|q\|^2_{0,\Omega} \right).
\end{aligned}$$

By the triangle inequality and (2.15), we have

$$\begin{aligned}
\|\nabla \cdot \mathbf{v}\|^2_{0,\Omega} \;&\le\; 2 \left(\|\nabla \cdot \mathbf{v} + Xq\|^2_{0,\Omega} + \|Xq\|^2_{0,\Omega} \right) \le 2 \left(\|\nabla \cdot \mathbf{v} + Xq\|^2_{0,\Omega} + C \, \|q\|^2_{1,\Omega} \right) \\
&\le\; C \left(G(\mathbf{v},\, q;\, 0) + \|q\|^2_{0,\Omega} \right).
\end{aligned}$$

Combining the above three inequalities yields (2.14).

With (2.14), we show the validity of (2.12) by the compactness argument. To this end, assume that (2.12) is not true. This implies that there exists a sequence $\{\mathbf{v}_n,\, q_n\} \in H_N(\mathrm{div};\,\Omega) \times V$ such that

$$\|\mathbf{v}_n\|^2_{H(\mathrm{div})} + \|q_n\|^2_{1,\,\Omega} = 1 \quad \text{and} \quad G(\mathbf{v},\, q;\, 0) \le \frac{1}{n} \tag{2.16}$$

Since V is compactly contained in $L^2(\Omega)$, there exists a subsequence $\{p_{n_k}\} \in V$ which converges in $L^2(\Omega)$. For any $k,\ l$ and $(\mathbf{v}_{n_k},\, p_{n_k}),\, (\mathbf{v}_{n_l},\, p_{n_l}) \in H_N(\mathrm{div};\,\Omega) \times V$, it follows from (2.14) and the triangle inequality that

$$\begin{aligned} & \|\mathbf{v}_{n_k} - \mathbf{v}_{n_l}\|^2_{H(\mathrm{div})} + \|q_{n_k} - q_{n_l}\|^2_{1,\,\Omega} \\ \le\ & C\left(G(\mathbf{v}_{n_k} - \mathbf{v}_{n_l},\, q_{n_k} - q_{n_l};\, 0) + \|q_{n_k} - q_{n_l}\|^2_{0,\,\Omega}\right) \\ \le\ & C\left(G(\mathbf{v}_{n_k},\, q_{n_k};\, 0) + G(\mathbf{v}_{n_l},\, q_{n_l};\, 0) + \|q_{n_k} - q_{n_l}\|^2_{0,\,\Omega}\right) \to 0. \end{aligned}$$

which implies that $(\mathbf{v}_{n_k},\, p_{n_k})$ is a Cauchy sequence in the complete space $H_N(\mathrm{div};\,\Omega) \times V$. Hence, there exists $(\mathbf{v},\, p) \in H_N(\mathrm{div};\,\Omega) \times V$ such that

$$\lim_{k\to\infty}\left(\|\mathbf{v}_{n_k} - \mathbf{v}\|_{H(\mathrm{div})} + \|q_{n_k} - q\|_{1,\Omega}\right) = 0.$$

Next, we show that

$$q = 0 \quad \text{and} \quad \mathbf{v} = \mathbf{0} \tag{2.17}$$

which contradict with (2.16) that

$$0 = \|\mathbf{v}\|^2_{H(\mathrm{div})} + \|q\|^2_{1,\,\Omega} = \lim_{k\to\infty} \|\mathbf{v}_{n_k}\|^2_{H(\mathrm{div})} + \|q_{n_k}\|^2_{1,\,\Omega} = 1.$$

To this end, for any $\phi \in V$, integration by parts and the Cauchy-Schwarz inequality give

$$\begin{aligned} a(q_{n_k},\, \phi) &= (A\nabla q_{n_k},\, \nabla\phi) + (Xq_{n_k},\, \phi) = (A\nabla q_{n_k} + \mathbf{v}_{n_k},\, \nabla\phi) + (Xq_{n_k} + \nabla\cdot\mathbf{v}_{n_k},\, \phi) \\ &\le G(\mathbf{v}_{n_k},\, q_{n_k};\, 0)^{\frac{1}{2}} \|\phi\|_{1,\Omega}. \end{aligned}$$

Since $\lim q_{n_k} = q$ in V, we then have

$$|a(q,\, \phi)| = \lim_{k\to\infty} |a(q_{n_k},\, \phi)| \le \lim_{k\to\infty} G(\mathbf{v}_{n_k},\, q_{n_k};\, 0)^{\frac{1}{2}} \|\phi\|_{1,\Omega} = 0.$$

Because (2.3) has a unique solution, we have that

$$q = 0.$$

Now, $\mathbf{v} = \mathbf{0}$ follows from (2.14):

$$\|\mathbf{v}\|^2_{H(\mathrm{div})} = \lim_{k\to\infty} \|\mathbf{v}_{n_k}\|^2_{H(\mathrm{div})} \le C \lim_{k\to\infty} \left(G(\mathbf{v}_{n_k},\, q_{n_k};\, 0) + \|q_{n_k}\|^2_{0,\,\Omega}\right) = 0.$$

This completes the proof of (2.17) and, hence, the theorem. ■

2.3 Inverse Norm Least-Squares Functional

For any $(\mathbf{v},\, q) \in H_N(\text{div};\, \Omega) \times V \equiv \Phi$, consider the following least-squares functional:

$$G(\mathbf{v},\, q;\, f) = \|A^{-\frac{1}{2}}(\mathbf{v} + A\nabla q)\|_{0,\Omega}^2 + \|\nabla \cdot \mathbf{v} + Xq - f\|_{-1,\,D}^2. \tag{2.18}$$

Let

$$\mathcal{K} = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & (-\Delta_D)^{-1} \end{pmatrix} \tag{2.19}$$

where $(-\Delta_D)^{-1}$ is the solution operator of the Laplace equation with homogeneous Dirichlet boundary conditions on Γ_D. Then the corresponding normal operator is

$$\mathcal{L}^*\,\mathcal{K}\,\mathcal{L} = \begin{pmatrix} A^{-1} - \nabla\,(-\Delta_D)^{-1}\,\text{div} & \nabla\,(I - (-\Delta_D)^{-1}\,X) \\ -(I - X^*\,(-\Delta_D)^{-1})\,\text{div} & -\text{div}\,A\,\nabla + X^*\,(-\Delta_D)^{-1}\,X \end{pmatrix} \tag{2.20}$$

with $\mathcal{L}$ defined in (2.4) and the corresponding bilinear and linear forms are

$$\begin{aligned} b(\mathbf{u},\, p;\, \mathbf{v},\, q) &= (A^{-1}(\mathbf{u} + A\nabla p),\, (\mathbf{v} + A\nabla q)) \\ &\qquad +((-\Delta_D)^{-1}(\nabla \cdot \mathbf{u} + Xp),\, \nabla \cdot \mathbf{v} + Xq) \end{aligned} \tag{2.21}$$

$$f(\mathbf{v},\, q) = ((-\Delta_D)^{-1} f,\, \nabla \cdot \mathbf{v} + Xq), \tag{2.22}$$

respectively.

Theorem 2.2 *There exist positive constants* α_0 *and* α_1 *such that*

$$\alpha_0 \left(\|\mathbf{v}\|^2 + \|q\|_{1,\,\Omega}^2 \right) \le b(\mathbf{v},\, q;\, \mathbf{v},\, q) = G(\mathbf{v},\, q;\, 0) \tag{2.23}$$

for any $(\mathbf{v},\, q) \in H_N(\text{div};\, \Omega) \times V$ *and*

$$b(\mathbf{u},\, p;\, \mathbf{v},\, q) \le \alpha_1 \left(\|\mathbf{u}\|^2 + \|p\|_{1,\,\Omega}^2 \right)^{\frac{1}{2}} \left(\|\mathbf{v}\|^2 + \|q\|_{1,\,\Omega}^2 \right)^{\frac{1}{2}} \tag{2.24}$$

for any $(\mathbf{u},\, p)$ *and any* $(\mathbf{v},\, q)$ *in* $H_N(\text{div};\, \Omega) \times V$.

Proof: The theorem may be proved in a similar fashion as that of Thereom 2.1. ■

This theorem gives the following equivalence:

$$\mathcal{L}^*\,\mathcal{K}\,\mathcal{L} \sim \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -\Delta \end{pmatrix}.$$

2.4 Div-Curl Least-Squares Functional

We use the following space to define the div-curl least-squares functional for the extended system (2.6). Let

$$H(\mathrm{curl}\, A;\, \Omega) = \{\mathbf{v} \in L^2(\Omega)^d \,:\, \nabla\times (A^{-1}\mathbf{v}) \in L^2(\Omega)^{2d-3}\}, \tag{2.25}$$

which is a Hilbert space under the norm

$$\|\mathbf{v}\|_{H(\mathrm{curl}\, A)} \equiv \left(\|\mathbf{v}\|^2_{0,\,\Omega} + \|\nabla\times (A^{-1}\mathbf{v})\|^2_{0,\,\Omega}\right)^{\frac{1}{2}}.$$

When A is the identity matrix in (2.25), we use the simpler notation $H(\mathrm{curl}; \Omega)$. Define the subspaces

$$H_D(\mathrm{curl}\, A;\, \Omega) = \{\mathbf{v} \in H(\mathrm{curl}\, A;\, \Omega) \,:\, \mathbf{n} \times (A^{-1}\mathbf{v}) = \mathbf{0} \text{ on } \Gamma_D\},$$

and

$$\mathbf{W} = H_N(\mathrm{div};\, \Omega) \cap H_D(\mathrm{curl}\, A;\, \Omega).$$

For $(\mathbf{v},\, q) \in \mathbf{W} \times V = \Phi$, the div-curl least-squares functional is given by

$$G(\mathbf{v},\, q;\, f) = \|A^{-\frac{1}{2}}(\mathbf{v} + A\nabla q)\|^2_{0,\,\Omega} + \|\nabla\cdot \mathbf{v} + Xq - f\|^2_{0,\,\Omega} + \|\nabla\times (A^{-1}\mathbf{v})\|^2_{0,\,\Omega}. \tag{2.26}$$

The corresponding normal operator is

$$\mathcal{L}^*\,\mathcal{L} = \begin{pmatrix} A^{-1} - \nabla\,\mathrm{div} + A^{-1}\,\nabla\times\,\nabla\times\,A^{-1} & \nabla\,(I - X) \\ -(I - X^*)\,\mathrm{div} & -\mathrm{div}\,A\,\nabla + X^*\,X \end{pmatrix} \tag{2.27}$$

with $\mathcal{L}$ defined in (2.6) and the corresponding normal operator and bilinear and linear forms are

$$\begin{aligned} b(\mathbf{u},\, p;\, \mathbf{v},\, q) &= (A^{-1}(\mathbf{u} + A\nabla p),\, \mathbf{v} + A\nabla q) + (\nabla\cdot\mathbf{u} + Xp,\, \nabla\cdot\mathbf{v} + Xq) \\ &\quad +(\nabla\times (A^{-1}\mathbf{u}),\, \nabla\times (A^{-1}\mathbf{v})) \end{aligned} \tag{2.28}$$

$$f(\mathbf{v},\, q) = (f,\, \nabla\cdot\mathbf{v} + Xq), \tag{2.29}$$

respectively. It follows from Theorem 2.1 that we have the following equivalence:

$$\mathcal{L}^*\,\mathcal{L} \sim \begin{pmatrix} I - \nabla\,\mathrm{div} + A^{-1}\,\nabla\times\,\nabla\times\,A^{-1} & \mathbf{0} \\ \mathbf{0} & -\Delta \end{pmatrix} \sim \begin{pmatrix} -\Delta & \mathbf{0} \\ \mathbf{0} & -\Delta \end{pmatrix}. \tag{2.30}$$

The second equivalence requires sufficient smoothness of coefficients and boundary (see [7] for the proof).

Theorem 2.3 *There exist positive constants* α_0 *and* α_1 *such that*

$$\alpha_0 \left(\|\mathbf{v}\|^2_{H(\mathrm{div})} + \|\nabla\times (A^{-1}\mathbf{v})\|^2_{0,\,\Omega} + \|q\|^2_{1,\,\Omega}\right) \le b(\mathbf{v},\, q;\, \mathbf{v},\, q) \tag{2.31}$$

for any $(\mathbf{v},\, q) \in \mathbf{W} \times V$ *and*

$$\begin{aligned} b(\mathbf{u},\, p;\, \mathbf{v},\, q) \;\; \le \;\; & \alpha_1 \left(\|\mathbf{u}\|^2_{H(\mathrm{div})} + \|\nabla\times (A^{-1}\mathbf{u})\|^2_{0,\,\Omega} + \|p\|^2_{1,\,\Omega}\right)^{\frac{1}{2}} \\ & \cdot \left(\|\mathbf{v}\|^2_{H(\mathrm{div})} + \|\nabla\times (A^{-1}\mathbf{v})\|^2_{0,\,\Omega} + \|q\|^2_{1,\,\Omega}\right)^{\frac{1}{2}} \end{aligned} \tag{2.32}$$

for any $(\mathbf{u},\, p),\; (\mathbf{v},\, q) \in \mathbf{W} \times V$.

2.5 Least-Squares Problems

For the solution space Φ, we have the following equivalent least-squares problems:
minimization problem: find $(\mathbf{u},\, p) \in \Phi$ such that

$$G(\mathbf{u},\, p;\, f) = \min_{(\mathbf{v},\, q)\in\Phi} G(\mathbf{v},\, q;\, f); \tag{2.33}$$

variational problem: find $(\mathbf{u},\, p) \in \Phi$ such that

$$b(\mathbf{u},\, p;\, \mathbf{v},\, q) = f(\mathbf{v},\, q) \quad \forall\; (\mathbf{v},\, q) \in \Phi. \tag{2.34}$$

2.6 Least-Squares Approximation

In this subsection, we consider least-squares finite element approximation only based on the div least-squares functional. Approximation based on the div-curl least-squares functional may be studied in a similar fashion. There are two numerical approximations based on the inverse norm least-squares functional: (1) mesh-dependent norm approach in [1] and (2) the discrete H^{-1} norm approach in [4].

Assume that Ω is a polygonal domain, let $\mathcal{T}_h$ be a quasi-regular triangulation of Ω with (triangular/tetrahedra or rectangular) elements of size $O(h)$. Denote spaces of polynomials on an element $K \subset \mathcal{R}^d$:

$P_k(K)$ is the space of polynomials of degree $\le k$;

$$P_{k_1,k_2}(K) = \{p(x_1,\, x_2) \,:\, p(x_1,\, x_2) = \sum_{i\le k_1,\; j\le k_2} a_{ij} x_1^i x_2^j, \qquad d=2$$

$$P_{k_1,k_2,k_3}(K) = \{p(x_1,\, x_2,\, x_3) \,:\, p(x_1,\, x_2,\, x_3) = \sum_{i\le k_1,\; j\le k_2,\; k\le k_3} a_{ijk} x_1^i x_2^j x_3^k, \qquad d=3.$$

Denote the local Raviart-Thomas (RT) space of index $k \ge 0$ on an element K:

$$RT_k(K) = \begin{cases} P_k(K)^d + (x_1, ..., x_d)P_k(K), & K = \text{triangle/tetrahedra} \\ P_{k+1,k}(K) \times P_{k,k+1}(K), & K = \text{rectangle },\, d=2 \\ P_{k+1,k,k}(K) \times P_{k,k+1,k}(K) \times P_{k,k,k+1}(K), & K = \text{rectangle },\, d=3 \end{cases}$$

Degrees of freedom for $RT_0(K) = (a + bx_1, c + bx_2)$ on triangle or $RT_0(K) = (a+bx_1, c+dx_2)$ on rectangle are normal components of vector field on all edges (faces) of two- (three-) dimensional elements. See [5] for the choice of degrees of freedom for the RT_k space of index $k \ge 1$. They are chosen for ensuring continuity of the normal component of vector field at interfaces of elements. Then one can define the $H(\text{div}; \Omega)$ conforming Raviart-Thomas space of order $k \ge 0$ [17] by

$$RT_k = \{\mathbf{v} \in H(\text{div}; \Omega) \,:\, \mathbf{v}|_K \in RT_k(K) \;\forall\; K \in \mathcal{T}_h\},$$

which has the approximation property:

$$\inf_{\boldsymbol{\phi} \in RT_k} \|\mathbf{v} - \boldsymbol{\phi}\|_{0,\Omega} \le C\, h^r \|\mathbf{v}\|_{r,\Omega} \quad \text{for} \quad 1 \le r \le k+1 \tag{2.35}$$

$$\inf_{\boldsymbol{\phi} \in RT_k} \|\nabla \cdot (\mathbf{v} - \boldsymbol{\phi})\|_{0,\Omega} \le C\, h^r \|\nabla \cdot \mathbf{v}\|_{r,\Omega} \quad \text{for} \quad 0 \le r \le k+1. \tag{2.36}$$

Denote the space of continuous piecewise polynomials of degree $\le k$ by

$$S_k = \{q \in H^1(\Omega) \;:\; q|_K \in P_k(K) \;\forall T \in \mathcal{T}_h\}.$$

which has the following approximation property:

$$\inf_{\phi \in S_k} \left(\|q - \phi\|_{0,\Omega} + h\,\|q - \phi\|_{1,\Omega}\right) \le C\, h^{r+1} \|q\|_{r+1,\Omega} \quad \text{for} \quad 0 \le r \le k+1. \tag{2.37}$$

Then least-squares approximation is to find $(\mathbf{u}^h,\, p^h) \in RT_k \times S_k$ such that

$$b(\mathbf{u}^h,\, p^h;\, \mathbf{v},\, q) = f(\mathbf{v},\, q) \quad \forall\; (\mathbf{v},\, q) \in RT_k \times S_k. \tag{2.38}$$

Theorem 2.4 *Let* $(\mathbf{u},\, p)$ *and* $(\mathbf{u}^h,\, p^h)$ *be the solutions of* (2.34) *and* (2.38), *respectively. Then we have the following error estimation:*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}^h\|_{H(\text{div})} + \|p - p^h\|_{1,\Omega} &\le C\frac{\alpha_1}{\alpha_0}\, h^r \left(\|p\|_{r+1,\Omega} + \|\mathbf{u}\|_{r,\Omega} + \|\nabla \cdot \mathbf{u}\|_{r,\Omega}\right) \\ &\le C\frac{\alpha_1}{\alpha_0}\, h^r \left(\|p\|_{r+1,\Omega} + \|f\|_{r,\Omega}\right). \end{aligned} \tag{2.39}$$

Proof: (2.39) follows from Theorem 1.1, the approximation properties in (2.35), (2.36), and (2.37), and the facts that

$$\|\mathbf{u}\|_{r,\Omega} \le C\, \|p\|_{r+1,\Omega}$$

and that

$$\|\nabla \cdot \mathbf{u}\|_{r,\Omega} = \|f - Xp\|_{r,\Omega} \le \|f\|_{r,\Omega} + C\, \|p\|_{r+1,\Omega}.$$

■

2.7 Comparison of Least-Squares Methods

In this section, we make simple comparison of least-squares methods. The div least-squares method has the following numerical properties:

+ optimal finite element approximation;

+ optimal fast multigrid solver if Raviart-Thomas elements are used for the flux.

The mesh dependent least-squares method has the following properties:

+ optimal finite element approximation;

− unknown fast iterative solver.

The discrete H^{-1} norm least-squares method has the following properties:

+ optimal finite element approximation;

+ uniformly well preconditioned by multigrid or domain decomposition;

− expensive evaluations of the discrete H^{-1} norm.

The div-curl least-squares method has the following properties:

+ finite element approximations are H^1-optimally accurate in each variable (including new variables);

+ *standard* multigrid methods applied to the resulting discrete equation have optimal complexity;

− additional smoothness of the original problem is required for the second equivalence in (2.30).

2.8 Boundary Least-Squares Functional

Denote by $H^{-\frac{1}{2}}(\partial\Omega)$ the dual space of $H^{\frac{1}{2}}(\partial\Omega)$ with the dual norm

$$\|v\|_{-\frac{1}{2},\partial\Omega} = \sup_{q\in H^{\frac{1}{2}}(\partial\Omega)} \frac{< v,\, q >}{\|q\|_{\frac{1}{2},\partial\Omega}},$$

where the bracket $< q,\, v >$ denotes duality between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$. When $\Gamma_N \neq \partial\Omega = \Gamma_N \cup \Gamma_D$, denote by $H^{-\frac{1}{2}}(\Gamma_N)$ the dual space of $H_{00}^{\frac{1}{2}}(\Gamma_N) = \{v|_{\Gamma_N} \,:\, v \in H^1_{0,D}(\Omega)\}$. In this section, we need the generalized Poincaré-Friedrichs inequality

$$\begin{aligned} &\|q\|_{1,\Omega} \le C\,(\|\nabla q\|_{0,\Omega} + \|q\|_{0,\Gamma_D}) && \forall\, q \in H^1(\Omega) && \text{if} \quad \text{mes}(\Gamma_D) \neq 0, \\ &\|q\|_{1,\Omega} \le C\,\|\nabla q\|_{0,\Omega} && \forall\, q \in \hat{H}^1(\Omega) && \text{otherwise}; \end{aligned} \tag{2.40}$$

and the trace inequalities for any subset $\Gamma \subset \partial\Omega$ with positive measure

$$\begin{aligned} \|q\|_{\frac{1}{2},\Gamma} &\le \|q\|_{1,\Omega} && \forall\, q \in H^1(\Omega), \\ \|\mathbf{n}\cdot\mathbf{v}\|_{-\frac{1}{2},\Gamma} &\le \|\mathbf{v}\|_{H(\mathrm{div})} && \forall\, \mathbf{v} \in H(\mathrm{div};\,\Omega). \end{aligned} \tag{2.41}$$

The first inequality in (2.41) follows from the definition. The second inequality in (2.41) follows from the definition, the Green's formula, and the Cauchy-Schwarz inequality: for any $q \in H^1(\Omega)$ and $q = 0$ on $\Gamma' = \partial\Omega \setminus \Gamma$

$$\frac{|\int_\Gamma q\,\mathbf{n}\cdot\mathbf{v}\,ds|}{\|q\|_{\frac{1}{2},\partial\Omega}} \le \frac{|\int_{\partial\Omega} q\,\mathbf{n}\cdot\mathbf{v}\,ds|}{\|q\|_{1,\Omega}} = \frac{|\int_\Omega q\,\nabla\cdot\mathbf{v}\,dx + \int_\Omega \mathbf{v}\cdot\nabla q\,dx|}{\|q\|_{1,\Omega}} \le \|\mathbf{v}\|_{H(\mathrm{div})}.$$

Considering non-homogeneous boundary conditions:

$$p = g \quad \text{on } \Gamma_D \quad \text{and} \quad -\,\mathbf{n}\cdot A\nabla\, p = h \quad \text{on } \Gamma_N$$

and the following least-squares functional:

$$\begin{aligned} G(\mathbf{v},\, q;\, \hat{f}) &= \|A^{-\frac{1}{2}}(\mathbf{v} + A\nabla q)\|^2_{0,\,\Omega} + \|\nabla\cdot\mathbf{v} + Xq - f\|^2_{0,\,\Omega} \\ &\quad + \|p - g\|^2_{\frac{1}{2},\Gamma_D} + \|\mathbf{n}\cdot\mathbf{v} - h\|^2_{-\frac{1}{2},\Gamma_N} \end{aligned} \tag{2.42}$$

for $(\mathbf{v},\, q) \in H(\mathrm{div};\,\Omega) \times H^1(\Omega)$, where $\hat{f} = (f,\, g,\, h)$. Then the least-squares problem for (2.4) is to minimize this quadratic functional over $H(\mathrm{div};\,\Omega)\times H^1(\Omega)$: find $(\mathbf{u},\, p) \in H(\mathrm{div};\,\Omega) \times H^1(\Omega)$ such that

$$G(\mathbf{u},\, p;\, \hat{f}) = \inf_{(\mathbf{v},\,q)\in H(\mathrm{div};\,\Omega)\times H^1(\Omega)} G(\mathbf{v},\, q;\, \hat{f}). \tag{2.43}$$

It is easy to see that the variational form for (2.43) is to find $(\mathbf{u},\, p) \in H(\mathrm{div};\,\Omega)\times H^1(\Omega)$ such that

$$b(\mathbf{u}_,\, p;\, \mathbf{v},\, q) = f(\mathbf{v},\, q), \quad \forall\,(\mathbf{v},\, q) \in H(\mathrm{div};\,\Omega) \times H^1(\Omega), \tag{2.44}$$

where the bilinear form $b(\cdot\,;\,\cdot) : (H(\mathrm{div};\,\Omega) \times H^1(\Omega))^2 \to \Re$ is defined by

$$\begin{aligned} b(\mathbf{u},\, p;\, \mathbf{v},\, q) &= (A^{-1}(\mathbf{u} + A\nabla p),\, \mathbf{v} + A\nabla q)_{0,\,\Omega} + (\nabla\cdot\mathbf{u} + Xp,\, \nabla\cdot\mathbf{v} + Xq)_{0,\,\Omega} \\ &+ \ < p,\, q >_{\frac{1}{2},\Gamma_D} + < \mathbf{n}\cdot\mathbf{u},\, \mathbf{n}\cdot\mathbf{v} >_{-\frac{1}{2},\Gamma_N} \end{aligned}$$

and the linear form $f(\cdot,\,\cdot) : H(\mathrm{div};\,\Omega) \times H^1(\Omega) \to \Re$ is defined by

$$f(\mathbf{v},\, q) = (f,\, \nabla\cdot\mathbf{v} + Xq)_{0,\,\Omega} + < g,\, q >_{\frac{1}{2},\Gamma_D} + < h,\, \mathbf{n}\cdot\mathbf{v} >_{-\frac{1}{2},\Gamma_N}.$$

Theorem 2.5 *Then there exist positive constants* α_0 *and* α_1 *such that*

$$\alpha_0 \left(\|\mathbf{v}\|^2_{H(\mathrm{div})} + \|q\|^2_{1,\,\Omega}\right) \le b(\mathbf{v},\, q;\, \mathbf{v},\, q) \tag{2.45}$$

for any $(\mathbf{v},\, q) \in H(\mathrm{div};\,\Omega) \times H^1(\Omega)$ *and*

$$b(\mathbf{u},\, p;\, \mathbf{v},\, q) \le \alpha_1 \left(\|\mathbf{u}\|^2_{H(\mathrm{div})} + \|p\|^2_{1,\,\Omega}\right)^{\frac{1}{2}} \left(\|\mathbf{v}\|^2_{H(\mathrm{div})} + \|q\|^2_{1,\,\Omega}\right)^{\frac{1}{2}} \tag{2.46}$$

for any $(\mathbf{u},\, p),\ (\mathbf{v},\, q) \in H(\mathrm{div};\,\Omega) \times H^1(\Omega)$.

Proof: The continuity of the bilinear form $b(\cdot\,;\,\cdot)$ in (2.46) is an immediate consequence of the Cauchy-Schwarz and trace inequalities. To show the validity of the coercivity of the bilinear form in (2.45), it suffices to prove that

$$\|\mathbf{v}\|^2_{H(\mathrm{div})} + \|q\|^2_{1,\Omega} \le C\,\left(b(\mathbf{v},\, q;\, \mathbf{v},\, q) + \|q\|^2_{0,\Omega}\right) \tag{2.47}$$

because (2.45) then follows from a standard compactness argument (see the proof of Theorem 2.1). To this end, first note that, using the triangle and trace inequalities,

$$\begin{aligned}
&|\int_{\partial\Omega} q\,\mathbf{n}\cdot\mathbf{v}\,ds| \le |\int_{\Gamma_D} q\,\mathbf{n}\cdot\mathbf{v}\,ds| + |\int_{\Gamma_N} q\,\mathbf{n}\cdot\mathbf{v}\,ds|\\
\le\;& \|q\|_{\frac12,\Gamma_D}\|\mathbf{n}\cdot\mathbf{v}\|_{-\frac12,\Gamma_D} + \|q\|_{\frac12,\Gamma_N}\|\mathbf{n}\cdot\mathbf{v}\|_{-\frac12,\Gamma_N}\\
\le\;& \|q\|_{\frac12,\Gamma_D}\|\mathbf{v}\|_{H(\mathrm{div})} + \|q\|_{1,\Omega}\|\mathbf{n}\cdot\mathbf{v}\|_{-\frac12,\Gamma_N}\\
\le\;& \|q\|_{\frac12,\Gamma_D}\left(\|\mathbf{v}\|_{0,\Omega} + \|\nabla\cdot\mathbf{v} + Xq\|_{0,\Omega} + \|Xq\|_{0,\Omega}\right) + \|q\|_{1,\Omega}\|\mathbf{n}\cdot\mathbf{v}\|_{-\frac12,\Gamma_N}\\
\le\;& C\,\|q\|_{\frac12,\Gamma_D}\|A^{-\frac12}\mathbf{v}\|_{0,\Omega} + C\,\|A^{\frac12}\nabla q\|_{0,\Omega}\left(\|q\|_{\frac12,\Gamma_D} + \|\mathbf{n}\cdot\mathbf{v}\|_{-\frac12,\Gamma_N}\right) + b(\mathbf{v},\, q;\, \mathbf{v},\, q).
\end{aligned}$$

The triangle inequality gives that

$$\|A^{-\frac12}\mathbf{v}\|_{0,\Omega} \le \|A^{-\frac12}(\mathbf{v} + A\nabla q)\|_{0,\Omega} + \|A^{\frac12}\nabla q\|_{0,\Omega}, \tag{2.48}$$

which, together with the above inequality, implies that

$$|\int_{\partial\Omega} q\,\mathbf{n}\cdot\mathbf{v}\,ds| \le C\,\left(\|q\|_{\frac12,\Gamma_D} + \|\mathbf{n}\cdot\mathbf{v}\|_{-\frac12,\Gamma_N}\right)\|A^{\frac12}\nabla q\|_{0,\Omega} + C\,b(\mathbf{v},\, q;\, \mathbf{v},\, q). \tag{2.49}$$

It follows from integration by parts, the Cauchy-Schwarz and Poincaré-Friedrichs inequalities, and (2.49) that

$$\begin{aligned}
&\|A^{\frac12}\nabla q\|^2_{0,\Omega} = (A^{-\frac12}(A\nabla q + \mathbf{v}),\, A^{\frac12}\nabla q)_{0,\Omega} + (q,\nabla\cdot\mathbf{v})_{0,\Omega} - \int_{\partial\Omega} q\,\mathbf{n}\cdot\mathbf{v}\,ds\\
\le\;& \|A^{-\frac12}(A\nabla q + \mathbf{v})\|_{0,\Omega}\|A^{\frac12}\nabla q\|_{0,\Omega} + \|q\|_{0,\Omega}\|\nabla\cdot\mathbf{v}\|_{0,\Omega} - \int_{\partial\Omega} q\,\mathbf{n}\cdot\mathbf{v}\,ds\\
\le\;& \left(\|A^{-\frac12}(A\nabla q + \mathbf{v})\|_{0,\Omega} + \|\nabla\cdot\mathbf{v} + Xq\|_{0,\Omega} + \|\mathbf{n}\cdot\mathbf{v}\|_{-\frac12,\Gamma_N} + \|q\|_{\frac12,\Gamma_D}\right)\|A^{\frac12}\nabla q\|_{0,\Omega}\\
&\quad + \|q\|_{0,\Omega}\|Xq\|_{0,\Omega} + C\,b(\mathbf{v},\, q;\, \mathbf{v},\, q).
\end{aligned}$$

Hence,

$$\|Xq\|^2_{0,\Omega} \le C\,\|A^{\frac12}\nabla q\|^2_{0,\Omega} \le C\,\left(b(\mathbf{v},\, q;\, \mathbf{v},\, q) + \|q\|^2_{0,\Omega}\right).$$

Combining with (2.48) and the triangle inequality yields

$$\|A^{-\frac12}\mathbf{v}\|^2_{0,\Omega} + \|\nabla\cdot\mathbf{v}\|^2_{0,\Omega} \le C\,\left(b(\mathbf{v},\, q;\, \mathbf{v},\, q) + \|q\|^2_{0,\Omega}\right).$$

This completes the proof of (2.47) and, hence, theorem. ■

For numerical approach based on the boundary least-squares functional, see [18].

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Homework

Consider problem (2.1) with $Xp = \mathbf{b} \cdot \nabla p + cp$, study dependence of constants α_0 and α_1 in Theorem 2.1 on the diffusion coefficients A, convection coefficients $\mathbf{b}$, and reaction coefficient c for the following cases:

- $A = a(x)\, I$, $\mathbf{b} = \mathbf{0}$, and $c = 0$.
- $A = I$, $\mathbf{b} = b \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $c = 0$, where b is a constant.
- $A = I$, $\mathbf{b} = \mathbf{0}$, and $c = -\omega$, where $\omega > 0$ is a constant.