Available online at www.sciencedirect.com
SCIENCE DIRECT®
Comput. Methods Appl. Mech. Engrg. 195 (2006) 2635–2648
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

A finite element method using singular functions for Poisson equations: Mixed boundary conditions ☆

Zhiqiang Cai [a,*], Seokchan Kim [b], Sangdong Kim [c], Sooryun Kong [b]

[a] Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067, United States
[b] Department of Applied Mathematics, Changwon National University, Changwon 641-773, Republic of Korea
[c] Department of Mathematics, Kyungpook National University, Republic of Korea

Received 29 April 2004; received in revised form 14 April 2005; accepted 2 June 2005

Abstract

In [Z. Cai, S. Kim, A finite element method using singular functions for the Poisson equation: corner singularities, SIAM J. Numer. Anal. 39 (2001) 286–299], we proposed a new finite element method to compute singular solutions of Poisson equations on a polygonal domain with re-entrant angles. Singularities are eliminated and only the regular part of the solution that is in H^2 is computed. The stress intensity factor and the solution can be computed as a post-processing step. This method is extended to problems with crack singularities and to a higher-order method for smooth data in [Z. Cai, S. Kim, G. Woo, A finite element method using singular functions for the Poisson equation: crack singularities, Numer. Linear Algebra Appl. 9 (2002) 445–455]. In this paper, we study the Poisson equation with mixed boundary conditions. Examples with various singular points and numerical results are presented.

Keywords: Corner singularity; Finite element; Stress intensity factor; Mixed boundary condition

1. Introduction

Assume that $\Omega \subset \mathscr{R}^2$ is a polygonal domain. Let Γ_D and Γ_N be a partition of the boundary of Ω such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. For simplicity, assume that Γ_D is not empty (i.e., meas$(\Gamma_D) \neq 0$). Let v denote the outward unit vector normal to the boundary. For a given function $f \in L^2(\Omega)$, consider the Poisson equation with homogeneous mixed boundary conditions:

☆ This work was sponsored in part by KRF-2002-070-C00014 and by INT-0139053.
* Corresponding author.
E-mail addresses: zcai@math.purdue.edu (Z. Cai), sckim@sarim.changwon.ac.kr (S. Kim), skim@knu.ac.kr (S. Kim), ksr@changwon.ac.kr (S. Kong).

0045-7825/$ - see front matter
doi:10.1016/j.cma.2005.06.004

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_{\mathrm{D}}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_{\mathrm{N}}, \end{cases} \tag{1.1}$$

where Δ stands for the Laplacian operator. Solution of (1.1) has singular behavior near corners even when f is very smooth. Such singular behavior affects the accuracy of the finite element method throughout the whole domain.

Solutions of many elliptic boundary value problems on polygonal domains have a singular function representation: a linear combination of singular functions and the regular part of the solution (see (2.9) for more details). Coefficients of singular functions in this representation are called the stress intensity factors. This property has been explored in several ways to design accurate finite element methods in the presence of corner singularities. One approach is the so-called Singular Function Method (SFM) (see, e.g., [16]) that augments singular functions to both the trial and test spaces. However, the convergence of the stress intensity factors is sometimes very poor (see [13,19]). They usually must be corrected by a suitable post-processing. To overcome this difficulty, the so-called Dual Singular Function Method (DSFM) was introduced (see, e.g., [14,5,7,15]), that augments singular functions to the trial space and the corresponding dual singular functions to the test space. The DSFM was implemented as an iterative procedure which iterates back and forth between the singular function representation formula, the original equation, and the extraction formula (see (2.10)) for the stress intensity factors. This approach was extended to full multigrid versions in [8].

Recently in [9,11] we also use this property in order to calculate accurate finite element approximations to both the solution and the stress intensity factors. The loss of standard finite element approximation accuracy for elliptic boundary value problems with corner singularities is due to the non-smoothness of the solution. Therefore, it is natural to first approximate the regular part of the solution, and then compute the stress intensity factors and the solution. By using the dual singular functions and a particularly chosen cut-off function, we are able to deduce a well-posed variational problem for the regular part of the solution. Based on this variational problem, we showed that continuous piecewise linear finite element approximation on a quasi-uniform triangulation yields $\mathrm{O}(h)$ optimal accuracy in H^1. Also, we established $\mathrm{O}(h^{1+\frac{\pi}{\omega}})$ error bound for the solution in L^2 and for the stress intensity factors in the absolute value, where ω depends on the re-entrant angles of the domain Ω. Our numerical experiments in [10] seem to indicate that our approach achieves $\mathrm{O}(h^2)$ accuracy.

The problem for the regular part of the solution is no longer a ''nice'' Poisson equation. Instead, it is a Poisson equation perturbed by integral terms which are only non-zero on strips away from the corners. Because of such perturbation, the problem is non-symmetric and possibly indefinite. To solve non-symmetric algebraic equations arising from the discretization, it was shown in both theory and numerics in [10] that a standard multigrid method is very efficient. This is because the non-symmetric perturbation with pseudo-differential order of -1 is well-controlled by the Laplace operator whose pseudo-differential order is 2. The solution method adopted in [10] is a simple V-cycle multigrid method that uses an exact coarsest grid solver and smoothing operators depending only on the discrete Laplace operator.

Corner singularities can also be overcome by the method of local grid refinement (see, e.g., [1,2]). Using this method, the number of degrees of freedom is of order $\mathrm{O}(h^{-2})$ and the error of the computed stress intensity factor is of order $\mathrm{O}(h^2)$. This method also has the advantage that it does not require the knowledge of the exact forms of the singular functions. It only needs the knowledge of the *exponents* of the singular functions. However, it is more difficult in this approach to use fast multilevel solution techniques (see [18,20,21]) because the mesh sizes of fine grids decade exponentially.

The purpose of this paper is to extend results for the Poisson equations with Dirichlet boundary conditions in [9] to mixed boundary conditions. Singular functions for mixed boundary conditions differ from those for Dirichlet boundary conditions in both form and angles of corners. In Section 2, singular function representation of the solution for various boundary conditions is presented and a variational problem for regular part of the solution is derived. In Section 3, we introduce a finite element approximation and estimate its error bound. Finally, in Section 4, we present two examples with various singular functions and their numerical results.

We will use the standard notation and definitions for the Sobolev spaces $H^t(\Omega)$ for $t \geqslant 0$; the standard associated inner products are denoted by $(\cdot,\cdot)_{t,\Omega}$, and their respective norms and seminorms are denoted by $\|\cdot\|_{t,\Omega}$ and $|\cdot|_{t,\Omega}$. The space $L^2(\Omega)$ is interpreted as $H^0(\Omega)$, in which case the inner product and norm will be denoted by $(\cdot,\cdot)_\Omega$ and $\|\cdot\|_\Omega$, respectively. $H^1_{\mathrm{D}}(\Omega) = \{u \in H^1(\Omega) : \ u = 0 \text{ on } \Gamma_{\mathrm{D}}\}$.

2. Singular function representations

Let $\omega_1, \ldots, \omega_M$ be internal angles of Ω satisfying

$$\begin{aligned} &\pi/2 < \omega_j < 2\pi \quad \text{if boundary condition changes its type,} \\ &\pi < \omega_j < 2\pi \quad \text{otherwise} \end{aligned}$$

and denote by $\mathbf{v}_j$ $(j = 1, \ldots, M)$ the corresponding vertices. Let the polar co-ordinates (r_j, θ_j) be chosen at the vertex $\mathbf{v}_j$ so that the internal angle ω_j is spanned *counterclockwise* by two half-lines $\theta_j = 0$ and $\theta_j = \omega_j$. Below is a list of singular functions at $\mathbf{v}_j$ depending on boundary conditions:

- D/D If $\omega_j > \pi$, there is a singular function of the form

$$s_{j,1}(r_j, \theta_j) = r_j^{\frac{\pi}{\omega_j}} \sin\frac{\pi\theta_j}{\omega_j}; \tag{2.1}$$

- N/N If $\omega_j > \pi$, there is a singular function of the form

$$s_{j,1}(r_j, \theta_j) = r_j^{\frac{\pi}{\omega_j}} \cos\frac{\pi\theta_j}{\omega_j}; \tag{2.2}$$

- D/N If $\frac{\pi}{2} < \omega_j \leqslant \frac{3\pi}{2}$, there is a singular function of the form

$$s_{j,\frac{1}{2}}(r_j, \theta_j) = r_j^{\frac{\pi}{2\omega_j}} \sin\frac{\pi\theta_j}{2\omega_j}. \tag{2.3}$$

If $\frac{3\pi}{2} < \omega_j < 2\pi$, there are two singular functions of the form

$$s_{j,\frac{1}{2}}(r_j, \theta_j) = r_j^{\frac{\pi}{2\omega_j}} \sin\frac{\pi\theta_j}{2\omega_j} \quad \text{and} \quad s_{j,\frac{3}{2}}(r_j, \theta_j) = r_j^{\frac{3\pi}{2\omega_j}} \sin\frac{3\pi\theta_j}{2\omega_j}; \tag{2.4}$$

- N/D If $\frac{\pi}{2} < \omega_j \leqslant \frac{3\pi}{2}$, there is a singular function of the form

$$s_{j,\frac{1}{2}}(r_j, \theta_j) = r_j^{\frac{\pi}{2\omega_j}} \cos\frac{\pi\theta_j}{2\omega_j}. \tag{2.5}$$

If $\frac{3\pi}{2} < \omega_j < 2\pi$, there are two singular functions of the form

$$s_{j,\frac{1}{2}}(r_j, \theta_j) = r_j^{\frac{\pi}{2\omega_j}} \cos\frac{\pi\theta_j}{2\omega_j} \quad \text{and} \quad s_{j,\frac{3}{2}}(r_j, \theta_j) = r_j^{\frac{3\pi}{2\omega_j}} \cos\frac{3\pi\theta_j}{2\omega_j}. \tag{2.6}$$

Here, D/D and N/N mean that type of boundary conditions remains unchanged while D/N and N/D mean that type of boundary conditions changes passing the vertex $\mathbf{v}_j$. For convenience, we denote index set of singular functions by L_j. Hence,

$$L_j = \begin{cases} \{1\} & \text{for (2.1) and (2.2)},\\ \left\{\frac{1}{2}\right\} & \text{for (2.3) and (2.5)},\\ \left\{\frac{1}{2},\frac{3}{2}\right\} & \text{for (2.4) and (2.6)}. \end{cases}$$

It is easy to see that $s_{j,\frac{3}{2}} \in H^{1+\frac{3\pi}{2\omega_j}-\varepsilon}(\Omega)$, $s_{j,1} \in H^{1+\frac{\pi}{\omega_j}-\varepsilon}(\Omega)$, and $s_{j,\frac{1}{2}} \in H^{1+\frac{\pi}{2\omega_j}-\varepsilon}(\Omega)$ for any $\varepsilon > 0$. Hence, singular functions at the vertex $\mathbf{v}_j$ belong to either $H^{1+\frac{\pi}{\omega_j}-\varepsilon}(\Omega)$ for D/D and N/N vertex or $H^{1+\frac{\pi}{2\omega_j}-\varepsilon}(\Omega)$ for D/N and N/D vertex. This indicates that the solution of Poisson equation (1.1) is in $H^{1+\frac{\pi}{\omega}-\varepsilon}$ where $\omega = \max_{1\leqslant j\leqslant M}\hat{\omega}_j$ and

$$\hat{\omega}_j = \begin{cases} \omega_j & \text{if } \mathbf{v}_j \text{ is D/D or N/N vertex},\\ 2\omega_j & \text{if } \mathbf{v}_j \text{ is D/N or N/D vertex}. \end{cases}$$

To deduce an equation for the regular part of the solution, we need to use the so-called dual singular functions that are defined as follows: for $l \in L_j$,

$$s_{j,-l}(r_j,\theta_j) = r_j^{-\frac{l\pi}{\omega_j}} \sin\frac{l\pi}{\omega_j}\theta_j \quad \text{and} \quad s_{j,-l}(r_j,\theta_j) = r_j^{-\frac{l\pi}{\omega_j}} \cos\frac{l\pi}{\omega_j}\theta_j, \tag{2.7}$$

are the dual singular functions corresponding to

$$s_{j,l}(r_j,\theta_j) = r_j^{\frac{l\pi}{\omega_j}} \sin\frac{l\pi}{\omega_j}\theta_j \quad \text{and} \quad s_{j,l}(r_j,\theta_j) = r_j^{\frac{l\pi}{\omega_j}} \cos\frac{l\pi}{\omega_j}\theta_j, \tag{2.8}$$

respectively. We will also need cut-off functions. To this end, set

$$B_j(t_1;t_2) = \left\{(r_j,\theta_j) : t_1 < r_j < t_2 \text{ and } 0 < \theta_j < \omega_j\right\} \cap \Omega \quad \text{and} \quad B_j(t_1) = B_j(0;t_1).$$

A family of cut-off functions of r_j, $\eta_{\rho_j}(r_j)$, is then defined as follows:

$$\eta_{\rho_j}(r_j) = \begin{cases} 1 & \text{in } B_j\left(\frac{1}{2}\rho_j R\right),\\ \frac{15}{16}\left\{\frac{8}{15} - \left(\frac{4r_j}{\rho_j R} - 3\right) + \frac{2}{3}\left(\frac{4r_j}{\rho_j R} - 3\right)^3 - \frac{1}{5}\left(\frac{4r_j}{\rho_j R} - 3\right)^5\right\} & \text{in } \overline{B}_j\left(\frac{1}{2}\rho_j R;\rho_j R\right),\\ 0 & \text{in } \Omega\setminus\overline{B}_j(\rho_j R), \end{cases}$$

where ρ_j is a parameter in $(0,2]$ and $R \in \mathscr{R}$ is a fixed number so that the $\eta_2 s_{j,l}$ has the same boundary condition as u. We assume that R is small enough so that the intersection of either $B_j(\rho_j R)$ and $B_i(2R)$ or $B_j(2R)$ and $B_i(\rho_i R)$ for $j \neq i$ is empty.

It is well known [3,12,14] that the solution of problem (1.1) has the following singular function representation:

$$u = w + \sum_{j=1}^{M}\sum_{l\in L_j}\lambda_{j,l}\eta_{\rho_j}(r_j)s_{j,l}(r_j,\theta_j), \tag{2.9}$$

where $w \in H^2(\Omega)\cap H^1_{\mathrm{D}}(\Omega)$ is the regular part of the solution and $\lambda_{j,l} \in \mathscr{R}$ are the stress intensity factors that can be expressed in terms of u by the following extraction formulas ([6,17]):

$$\lambda_{j,l} = \frac{1}{l\pi}\left(\int_\Omega f\eta_{\rho_j}s_{j,-l}\,\mathrm{d}x + \int_\Omega u\Delta(\eta_{\rho_j}s_{j,-l})\,\mathrm{d}x\right). \tag{2.10}$$

Moreover, the following regularity estimate holds:

$$\|w\|_2 + \sum_{j=1}^{M} \sum_{l \in L_j} |\lambda_{j,l}| \leqslant C_R \|f\|. \tag{2.11}$$

In the remainder of this section, we derive a well-posed problem for w. To this end, assume that ρ_j in (2.9) belongs to $(0,1]$ and denote cut-off functions with bigger supports by

$$\eta^*(r_j) = \eta_2(r_j).$$

Choosing $\eta_\rho(r_j) = \eta^*(r_j)$ in (2.10) gives

$$\lambda_{j,l} = \frac{1}{l\pi}(u, \Delta(\eta^* s_{j,-l})) + \frac{1}{l\pi}(f, \eta^* s_{j,-l}).$$

Substituting $u = w + \sum_{i=1}^{M}\sum_{k\in L_i} \lambda_{i,k}\eta_{\rho_i}(r_i)s_{i,k}(r_i,\theta_i)$ into the above equation yields

$$\lambda_{j,l} = \frac{1}{l\pi}(w, \Delta(\eta^* s_{j,-l})) + \frac{1}{l\pi}(f, \eta^* s_{j,-l}) + \frac{1}{l\pi}\sum_{i=1}^{M}\sum_{k\in L_i} \lambda_{i,k}(\eta_{\rho_i}s_{i,k}, \Delta(\eta^* s_{j,-l})). \tag{2.12}$$

When $i = j$, the support of $\eta_{\rho_j}(r_j)$ for $0 < \rho_j \leqslant 1$ is $B_j(\rho_j R)$ on which $\eta^* = 1$. Since $s_{j,-l}$ is harmonic, then for all $k \in L_j$,

$$(\eta_{\rho_i}s_{i,k}, \Delta(\eta^* s_{j,-l})) = 0.$$

When $i \neq j$, by the assumption that $B_i(\rho_i R) \cap B_j(2R) = \emptyset$ we have that

$$(\eta_{\rho_i}s_{i,k}, \Delta(\eta^* s_{j,-l})) = 0, \quad \forall k \in L_i.$$

Hence, we have established the following extraction formulas of $\lambda_{j,l}$ in terms of w:

$$\lambda_{j,l} = \frac{1}{l\pi}(w, \Delta(\eta^* s_{j,-l}))_{B_j(R;2R)} + \frac{1}{l\pi}(f, \eta^* s_{j,-l})_{B_j(2R)}. \tag{2.13}$$

Using (2.13) and substituting (2.9) into the Poisson equation, we obtain an integro-differential equation for w:

$$-\Delta w - \sum_{j=1}^{M}\sum_{l\in L_j} \frac{1}{l\pi}(w, \Delta(\eta^* s_{j,-l}))_{B_j(R;2R)}\Delta(\eta_{\rho_j}s_{j,l}) = f + \sum_{j=1}^{M}\sum_{l\in L_j}\frac{1}{l\pi}(f, \eta^* s_{j,-l})_{B_j(2R)}\Delta(\eta_{\rho_j}s_{j,l}) \quad \text{in } \Omega.$$

Multiplying the above equation by a test function $v \in H^1_{\mathrm{D}}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_{\mathrm{D}}\}$, integrating over Ω, and using integration by parts lead to the following variational problem: finding $w \in H^2(\Omega) \cap H^1_{\mathrm{D}}(\Omega)$ such that

$$a(w, v) = g(v) \quad \forall v \in H^1_{\mathrm{D}}(\Omega), \tag{2.14}$$

where the bilinear form $a(\cdot,\cdot)$ and linear form $g(\cdot)$ are defined by

$$\begin{aligned} &a(w,v) = a^s(w,v) + b(w,v), \quad a^s(w,v) = (\nabla w, \nabla v),\\ &b(w,v) = -\sum_{j=1}^{M}\sum_{l\in L_j}\frac{1}{l\pi}(w, \Delta(\eta^* s_{j,-l}))_{B_j(R;2R)}(\Delta(\eta_{\rho_j}s_{j,l}), v)_{B_j\left(\frac{1}{2}\rho_j R;\rho_j R\right)} \end{aligned} \tag{2.15}$$

and

$$g(v) = (f, v) + \sum_{j=1}^{M}\sum_{l\in L_j}\frac{1}{l\pi}(f, \eta^* s_{j,-l})_{B_j(2R)}(\Delta(\eta_{\rho_j}s_{j,l}), v)_{B_j\left(\frac{1}{2}\rho_j R;\rho_j R\right)}. \tag{2.16}$$

Note that the second terms in the respective bilinear and linear forms provide a singular correction so that $w \in H^2(\Omega)$ for $f \in L^2(\Omega)$. Note also that the bilinear forms $a(\cdot,\cdot)$ are not symmetric.

Lemma 2.1. *For any* $0 < \rho \leqslant 1$, *we have that*

$$\|\Delta(\eta^* s_{j,-l})\|_{B_j(R;2R)} \leqslant C_1 R^{-\frac{l\pi}{\omega_j}-1} \tag{2.17}$$

with $C_1 = \sqrt{\frac{120 l\pi}{7}}$ *and that*

$$\|\eta_{\rho_j} s_{j,l}\|_{B_j(\rho_j R)} \leqslant C_2(\rho_j R)^{1+\frac{l\pi}{\omega_j}} \quad \text{and} \quad \|\nabla(\eta_{\rho_j} s_{j,l})\|_{B_j(\rho_j R)} \leqslant C_3(\rho_j R)^{\frac{l\pi}{\omega_j}} \tag{2.18}$$

with $C_2 = \frac{\omega_j}{2\sqrt{l\pi+\omega_j}}$ *and* $C_3 = \left(\frac{C^2\omega_j^2}{4(l\pi+\omega_j)}(1-2^{-2(\frac{l\pi}{\omega_j}+1)}) + \frac{l\pi}{2}\right)^{\frac{1}{2}}$.

Proof. This lemma can be established by an elementary calculation. □

In a similar fashion as in [9], we can prove the coercivity and continuity of the bilinear form $a(\cdot,\cdot)$ and the well-posedness of problem (2.14).

Lemma 2.2. *For* $0 < \rho \leqslant 1$, *the bilinear forms* $a(\cdot,\cdot)$ *are continuous and coercive in* $H^1_{\rm D}(\Omega)$; *i.e. there exist positive constants* α, K_1, *and* K_2 *such that*

$$\alpha\|\phi\|_1^2 \leqslant a(\phi,\phi) + K_1\|\phi\|^2 \tag{2.19}$$

for all $\phi \in H^1_{\rm D}(\Omega)$ *and that*

$$a(\phi,\psi) \leqslant K_2\|\phi\|_1\|\psi\|_1 \tag{2.20}$$

for all ϕ *and* ψ *in* $H^1_{\rm D}(\Omega)$.

Theorem 2.1. *For* $0 < \rho \leqslant 1$, *we have that*

(1) *if* $f \in L^2(\Omega)$, *then problem* (2.14) *has a unique solution* $w \in H^2(\Omega) \cap H^1_{\rm D}(\Omega)$,
(2) *there exists a positive constant* γ *such that*

$$\gamma\|\phi\|_1 \leqslant \sup_{\psi \in H^1_{\rm D}(\Omega)} \frac{a(\phi,\psi)}{\|\psi\|_1} \tag{2.21}$$

for any $\phi \in H^1_{\rm D}(\Omega)$.

3. Finite element approximation

This section presents standard finite element approximation on a quasi-uniform grid for w based on the variational problem in (2.14). Approximations to the stress intensity factors and the solution of problem (1.1) can then be calculated according to (2.13) and (2.9), respectively. Error estimates are established in Theorem 3.1.

Let $\mathscr{T}_h$ be a partition of the domain Ω into triangular finite elements; i.e., $\Omega = \bigcup_{K\in\mathscr{T}_h} K$ with $h = \max\{\operatorname{diam} K : K \in \mathscr{T}_h\}$. Assume that the triangulation $\mathscr{T}_h$ is regular. Denote continuous piecewise linear finite element space by

$$V_h = \{\phi_h \in C^0(\Omega) : \phi_h|_K \text{ is linear } \forall K \in \mathscr{T}_h \text{ and } \phi_h = 0 \text{ on } \Gamma_{\rm D}\} \subset H^1_{\rm D}(\Omega).$$

It is well known that

$$\inf_{\phi_h\in V_h}(\|\phi-\phi_h\| + h|\phi-\phi_h|_1) \leqslant C_A h^{1+t}\|\phi\|_{1+t,\Omega} \tag{3.1}$$

for any $\phi \in H^1_{\mathrm{D}}(\Omega) \cap H^{1+t}(\Omega)$ and $0 \leqslant t \leqslant 1$. The finite element approximation to problem (2.14) is to find $w_h \in V_h$ such that

$$a(w_h, v) = g(v) \quad \forall v \in V_h. \tag{3.2}$$

Approximations to the $\lambda_{j,l}$ and the solution are calculated as follows:

$$\lambda^h_{j,l} = \frac{1}{l\pi}(w_h, \Delta(\eta^* s_{j,-l}))_{B_j(R;2R)} + \frac{1}{l\pi}(f, \eta^* s_{j,-l})_{B_j(2R)} \tag{3.3}$$

and

$$u_h = w_h + \sum_{j=1}^{M} \sum_{l \in L_j} \lambda^h_{j,l} \eta_{\rho_j}(r_j) s_{j,l}(r_j, \theta_j). \tag{3.4}$$

In order to establish the error bound in the L^2-norm, we consider the following adjoint problem of (2.14) with a simplified linear form: find $z \in H^1_{\mathrm{D}}(\Omega)$ such that

$$a(v, z) = (w - w_h, v) \quad \forall v \in H^1_{\mathrm{D}}(\Omega). \tag{3.5}$$

The next lemma establishes the well-posedness of problem (3.5) and provides the regularity estimate for z.

Lemma 3.1. *For $0 < \rho_j \leqslant 1$, problem (3.5) has a unique solution z in $H^1_{\mathrm{D}}(\Omega)$. Moreover, there is a singular function representation*

$$z = w_z + \sum_{j=1}^{M} \sum_{l \in L_j} \lambda^z_{j,l} \eta_{\rho_j} s_{j,l}, \tag{3.6}$$

where $w_z \in H^2(\Omega) \cap H^1_{\mathrm{D}}(\Omega)$ and $\lambda^z_{j,l} \in R$ satisfy the regularity estimate

$$\|w_z\|_2 + \sum_{j=1}^{M} \sum_{l \in L_j} |\lambda^z_{j,l}| \leqslant C'_R \|w - w_h\|. \tag{3.7}$$

Proof. Similar to Theorem 2.1, the adjoint problem in (3.5) has a unique solution in $H^1_{\mathrm{D}}(\Omega)$ and that there exists a positive constant γ' such that

$$\gamma' \|\psi\|_1 \leqslant \sup_{\phi \in H^1_{\mathrm{D}}(\Omega)} \frac{a(\phi, \psi)}{\|\phi\|_1} \quad \forall \psi \in H^1_{\mathrm{D}}(\Omega).$$

Let z be the solution of (3.5), by the Cauchy–Schwarz inequality we then have that

$$\|z\|_1 \leqslant \frac{1}{\gamma'} \sup_{\phi \in H^1_{\mathrm{D}}(\Omega)} \frac{a(\phi, z)}{\|\phi\|_1} = \frac{1}{\gamma'} \sup_{\phi \in H^1_{\mathrm{D}}(\Omega)} \frac{(w - w_h, \phi)}{\|\phi\|_1} \leqslant \frac{1}{\gamma'} \|w - w_h\|. \tag{3.8}$$

It is easy to check that the solution, $z \in H^1_{\mathrm{D}}(\Omega)$, of problem (3.5) satisfies

$$\Delta z = \sum_{j=1}^{M} \sum_{l \in L_j} \frac{1}{l\pi} (\nabla z, \nabla(\eta_{\rho_j} s_{j,l})) \Delta(\eta^* s_{j,-l}) - (w - w_h) \quad \text{in } \Omega. \tag{3.9}$$

Since the right-hand side of the above equation is at least in $L^2(\Omega)$, so is Δz. Therefore, z has the singular function representation

$$z = w_z + \sum_{j=1}^{M} \sum_{l \in L_j} \lambda^z_{j,l} \eta_{\rho_j} s_{j,l},$$

where $w_z \in H^2(\Omega) \cap H^1_{\mathrm{D}}$ and

$$\|w_z\|_2 + \sum_{j=1}^{M} \sum_{l \in L_j} |\lambda^z_{j,l}| \leqslant C_R \|\Delta z\|.$$

Now, the regularity bound in (3.7) follows from the triangle and Cauchy–Schwarz inequalities, (3.8), and Lemma 2.1 that

$$\begin{aligned}\|w_z\|_2 + \sum_{j=1}^{M} \sum_{l \in L_j} |\lambda^z_{j,l}| \leqslant C_R \|\Delta z\| &\leqslant C_R \left(\sum_{j=1}^{M} \sum_{l \in L_j} \frac{1}{l\pi} |(\nabla(\eta_{\rho_j} s_{j,l}), \nabla z)_{B(\rho_j R)}| \, \|\Delta(\eta^* s_{j,-l})\|_{B(R;2R)} + \|w - w_h\| \right) \\ &\leqslant C_R \left(\sum_{j=1}^{M} \sum_{l \in L_j} \frac{C_1 C_3}{\gamma' R l \pi} \rho_j^{\frac{l\pi}{\omega_j}} + 1 \right) \|w - w_h\|.\end{aligned}$$

This proves the inequality in (3.7) with

$$C'_R = C_R \left(\sum_{j=1}^{M} \sum_{l \in L_j} \frac{C_1 C_3}{\gamma' R l \pi} \rho^{\frac{l\pi}{\omega_j}} + 1 \right)$$

and, hence, the lemma. □

Now we are ready to establish error bounds for the finite element approximations.

Theorem 3.1

(i) *For $0 < \rho_j \leqslant 1$, there exists a positive constant h_0 such that for all $h \leqslant h_0$ (3.2) has a unique solution w_h in V_h. Moreover, let $w \in H^2(\Omega) \cap H^1_{\mathrm{D}}(\Omega)$ be the solution of (2.14), then we have the following error estimates*:

$$\|w - w_h\|_1 \leqslant C_4 h \|f\| \quad \textit{and} \quad \|w - w_h\| \leqslant C_5 h^{1+\frac{\pi}{\omega}} \|f\|. \tag{3.10}$$

(ii) *Let $\lambda_{j,l}$ and $\lambda^h_{j,l}$ be defined in (2.13) and (3.3), respectively. Then*

$$|\lambda_{j,l} - \lambda^h_{j,l}| \leqslant \frac{C}{l\pi} R^{-\frac{l\pi}{\omega_j} - 1} \|w - w_h\| \leqslant C_6 R^{-\frac{l\pi}{\omega_j} - 1} h^{1+\frac{\pi}{\omega}} \|f\|. \tag{3.11}$$

(iii) *Let u be the solution of (1.1) and u_h be its approximation defined in (3.4), then we have the following error estimates*:

$$\|u - u_h\|_1 \leqslant C_7 h \|f\| \quad \textit{and} \quad \|u - u_h\| \leqslant C_8 h^{1+\frac{\pi}{\omega}} \|f\|. \tag{3.12}$$

Proof. (i) We first establish error bounds in (3.10) for any solution to problem (3.2) that may exist. Then, for $f \equiv 0$, the uniqueness of the solution to problem (2.14) and the error bound in (3.10) imply that $w_h \equiv 0$. Hence, (3.2) has a unique solution w_h in V_h since it is a finite dimensional problem with the same number of unknowns and equations.

To establish error bounds, note first the orthogonality property

$$a(w - w_h, v) = 0 \quad \forall v \in V_h. \tag{3.13}$$

By choosing $v = w - w_h$ in (3.5) and using the orthogonality property in (3.13) and the continuity bound in (2.20), we have that

$$\|w - w_h\|^2 = a(w - w_h, z) = a(w - w_h, z - I_h z) \leqslant K_2 \|w - w_h\|_1 \|z - I_h z\|_1, \tag{3.14}$$

where $I_h z \in V_h$ is the nodal interpolant of z. From the triangle inequality, approximation property (3.1), the fact that (see [4])

$$\|\eta_{\rho_j} s_{j,l} - I_h(\eta_{\rho_j} s_{j,l})\|_1 \leqslant Ch^{\frac{l\pi}{\omega_j}}$$

and Lemma 3.1, one has

$$\begin{aligned}\|z - I_h z\|_1 &\leqslant \|w_z - I_h w_z\|_1 + \sum_{j=1}^{M}\sum_{l\in L_j} |\lambda^z_{j,l}|\,\|\eta_{\rho_j} s_{j,l} - I_h(\eta_{\rho_j} s_{j,l})\|_1 \leqslant Ch\|w_z\|_2 + \sum_{j=1}^{M}\sum_{l\in L_j} Ch^{\frac{l\pi}{\omega_j}}|\lambda^z_{j,l}| \\ &\leqslant C_{\mathrm{D}} h^{\frac{\pi}{\omega}}\|w - w_h\|.\end{aligned}$$

Substituting this into (3.14) and dividing $\|w - w_h\|$ on both sides give

$$\|w - w_h\| \leqslant K_2 C_{\mathrm{D}} h^{\frac{\pi}{\omega}}\|w - w_h\|_1. \tag{3.15}$$

Now, it follows Lemma 2.2, orthogonality property (3.13), and inequality (3.15) that for any $v \in V_h$

$$\begin{aligned}\alpha\|w - w_h\|_1^2 &\leqslant a(w - w_h, w - w_h) + K_1\|w - w_h\|^2 = a(w - w_h, w - v) + K_1\|w - w_h\|^2 \\ &\leqslant K_2\|w - w_h\|_1\|w - v\|_1 + K_1(K_2 C_{\mathrm{D}} h^{\frac{\pi}{\omega}})^2\|w - w_h\|_1^2,\end{aligned}$$

which, together with approximation property (3.1), implies the validity of the first error bound in (3.10) with $C_4 = 2\alpha^{-1}K_2 C_A C_R$ for all $h \leqslant h_0$. Here,

$$h_0 = \left(\frac{\alpha}{2K_1(K_2 C_{\mathrm{D}})^2}\right)^{\frac{\omega}{2\pi}}.$$

The second error bound in (3.10) is then a direct consequence of (3.15) with $C_5 = C_4 K_2 C_{\mathrm{D}}$.

(ii) Note from (2.13) and (3.3) that

$$\lambda_{j,l} - \lambda^h_{j,l} = \frac{1}{l\pi}(w - w_h, \Delta(\eta^* s_{j,-l}))_{B_j(R;2R)}.$$

Hence, (3.11) follows from the Cauchy–Schwarz inequality, Theorem 3.1(i), and Lemma 2.1 that

$$|\lambda_{j,l} - \lambda^h_{j,l}| \leqslant \frac{1}{l\pi}\|w - w_h\|\|\Delta(\eta^* s_{j,-l})\|_{B_j(R;2R)} \leqslant C_6 R^{-\frac{l\pi}{\omega_j}-1} h^{1+\frac{\pi}{\omega}}\|f\|$$

with $C_6 = \frac{C_5 C_1}{l\pi}$.

(iii) It follows from (2.9) and (3.4) that

$$u - u_h = (w - w_h) + \sum_{j=1}^{M}\sum_{l\in L_j}(\lambda_{j,l} - \lambda^h_{j,l})\eta_{\rho_j} s_{j,l}.$$

By using the triangle inequality, Lemma 2.1, (3.10) and (3.11), we have that

$$\begin{aligned}\|u - u_h\|_1 &\leqslant \|w - w_h\|_1 + \sum_{j=1}^{M}\sum_{l\in L_j}|\lambda_{j,l} - \lambda^h_{j,l}|\,\|\eta_{\rho_j} s_{j,l}\|_{1,B(\rho_j R)} \\ &\leqslant C_4 h\|f\| + \sum_{j=1}^{M}\sum_{l\in L_j} C_6 \rho_j^{\frac{l\pi}{\omega_j}}(C_2\rho + C_3 R^{-1}) h^{1+\frac{\pi}{\omega}}\|f\|.\end{aligned}$$

Therefore, the first inequality of (3.12) is valid with $C_7 = C_4 + C_6\sum_{j=1}^{M}\sum_{l\in L_j}\rho_j^{\frac{l\pi}{\omega_j}}(C_2\rho_j + C_3R^{-1})h^{\frac{\pi}{\omega}}$. In a similar fashion, by Lemma 2.1, (3.10) and (3.11), we may prove the validity of the second inequality of (3.12) with $C_8 = C_5 + C_6C_2\sum_{j=1}^{M}\sum_{l\in L_j}\rho_j^{1+\frac{l\pi}{\omega_j}}$. This completes the proof of the theorem. □

4. Numerical results

In this section, numerical results for Poisson equations with mixed boundary conditions are presented. One example is defined on the unit square and the other on a domain with re-entrance corner.

Example 1. Consider the Poisson equation in (1.1) with mixed boundary conditions on the unit square $\Omega = \{(x,y) \in \mathscr{R}^2 : 0 < x < 1, 0 < y < 1\}$. Homogeneous Neumann boundary is $\Gamma_{\mathrm{N}} = \{(x,0) \in \mathscr{R}^2 : 0 < x < 1/2\}$ and homogeneous Dirichlet boundary is $\Gamma_{\mathrm{D}} = \partial\Omega\backslash\Gamma_{\mathrm{N}}$ (see Fig. 1(a)). This problem has a geometric singularity at boundary point $(1/2, 0)$, where the boundary conditions change from Dirichlet to Neumann with an internal angle $\omega = \pi$. More specifically, the corresponding singular function has the form

$$s = r^{\frac{1}{2}}\sin\left(\frac{\theta}{2}\right).$$

Let η_2 be the cut-off function defined in Section 2 with $R = 1/4$ and choose the right-hand side function in (1.1) to be

$$f = -\sin(\pi x)[-\pi^2y^2(y-1) + 2(3y-1)] - \Delta(\eta_2 s).$$

Then the exact solution of the underlying problem is

$$u = w + \eta_\rho s,$$

where η_ρ is the cut-off function with $R = 1/4$ and $0 < \rho \leqslant 1$ and

$$w = \sin(\pi x)y^2(y-1) + (\eta_2 - \eta_\rho)s$$

is the regular part of the solution. Numerical results are presented in Tables 1–3, respectively, that confirm theoretical estimates.

Example 2. In this example, we consider a polygonal domain with a re-entrant corner (see Fig. 1(b)):

$$\Omega = \{(x,y) \in \mathscr{R}^2 : -1 < x < 1, -1 < y < 1\} \setminus \overline{T},$$

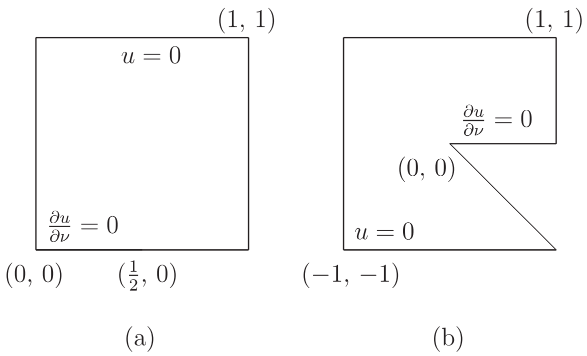

Fig. 1. Computational domains and boundary conditions. (a) Example 1 and (b) Example 2.

Table 1
The discrete L^2-norm errors and the convergence rates for w

Mesh size	$\rho = 1.00$		$\rho = 0.50$	
	L^2-norm	Rate	L^2-norm	Rate
$h = \frac{1}{8}$	2.3331e−02		2.0938e−02	
$h = \frac{1}{16}$	8.5794e−03	1.4433	6.8472e−03	1.6125
$h = \frac{1}{32}$	2.5658e−03	1.7414	2.1656e−03	1.6607
$h = \frac{1}{64}$	6.8989e−04	1.8949	5.9684e−04	1.8593
$h = \frac{1}{128}$	1.7708e−04	1.9619	1.5436e−04	1.9510

Table 2
The discrete H^1-seminorm and the convergence rates for w

	$\rho = 1.00$		$\rho = 0.50$	
	H^1-norm	Rate	H^1-norm	Rate
$h = \frac{1}{8}$	3.5511e−01		3.2662e−01	
$h = \frac{1}{16}$	1.7541e−01	1.0175	1.6750e−01	0.9635
$h = \frac{1}{32}$	5.2587e−02	1.7378	5.9165e−02	1.5013
$h = \frac{1}{64}$	1.4789e−02	1.8301	1.6557e−02	1.8373
$h = \frac{1}{128}$	3.9239e−03	1.9141	4.3852e−03	1.9167

Table 3
The absolute value errors and the convergence rates for λ

	$\rho = 1.00$		$\rho = 0.50$	
	$\lvert\lambda - \lambda_h\rvert$	Rate	$\lvert\lambda - \lambda_h\rvert$	Rate
$h = \frac{1}{8}$	6.8612e−01		6.6797e−01	
$h = \frac{1}{16}$	1.9996e−01	1.7787	1.9242e−01	1.7955
$h = \frac{1}{32}$	5.3705e−02	1.8965	5.3134e−02	1.8565
$h = \frac{1}{64}$	1.3776e−02	1.9628	1.6557e−02	1.8373
$h = \frac{1}{128}$	3.4729e−03	1.9879	4.3852e−03	1.9167

where $T = \{(x,y) \in \mathscr{R}^2 : 0 < x < 1, -x < y < 0\}$. Homogeneous Neumann boundary is $\Gamma_N = \{(x,0) \in \mathscr{R}^2 : 0 < x < 1, y = 0\}$ and homogeneous Dirichlet boundary is $\Gamma_D = \partial\Omega \backslash \Gamma_N$. At the origin, the internal angle is $\omega = \frac{7\pi}{4}$ and the boundary conditions change from Neumann to Dirichlet. Hence, there are two singular functions at the origin:

$$s_{\frac{1}{2}} = r^{\frac{2}{7}} \cos\frac{2\theta}{7} \quad \text{and} \quad s_{\frac{3}{2}} = r^{\frac{6}{7}} \cos\frac{6\theta}{7}.$$

Let η_ρ be the cut-off function with $R = 1/4$ and $0 < \rho \leqslant 1$ and choose the right-hand side function in (1.1) to be

$$f = f_1 - \Delta(\eta_2(s_{\frac{1}{2}} + s_{\frac{3}{2}})),$$

Table 4
The discrete L^2-norm errors and the convergence rates for w

Mesh size	$\rho = 1.00$		$\rho = 0.50$	
	L^2-norm	Rate	L^2-norm	Rate
$h = \frac{1}{8}$	1.0409e−01		7.5423e−02	
$h = \frac{1}{16}$	3.8311e−02	1.4420	2.5903e−02	1.5418
$h = \frac{1}{32}$	1.1322e−02	1.7586	8.4301e−03	1.6195
$h = \frac{1}{64}$	3.1228e−03	1.8582	2.3364e−03	1.8513

Table 5
The discrete H^1-seminorm and the convergence rates for w

	$\rho = 1.00$		$\rho = 0.50$	
	H^1-norm	Rate	H^1-norm	Rate
$h = \frac{1}{8}$	7.7863e−01		6.8214e−01	
$h = \frac{1}{16}$	3.4152e−01	1.1889	2.8441e−01	1.2621
$h = \frac{1}{32}$	9.9839e−02	1.7742	1.1067e−01	1.3617
$h = \frac{1}{64}$	2.7625e−02	1.8536	3.2023e−02	1.7891

Table 6
The absolute value errors and the convergence rates for $\lambda_{\frac{1}{2}}$

	$\rho = 1.00$		$\rho = 0.50$	
	$\lvert\lambda_{\frac{1}{2}} - \lambda_{\frac{1}{2},h}\rvert$	Rate	$\lvert\lambda_{\frac{1}{2}} - \lambda_{\frac{1}{2},h}\rvert$	Rate
$h = \frac{1}{8}$	6.6352e−01		6.5044e−01	
$h = \frac{1}{16}$	1.7794e−01	1.8987	1.7276e−01	1.9126
$h = \frac{1}{32}$	4.7506e−02	1.9052	4.7599e−02	1.8598
$h = \frac{1}{64}$	1.2370e−02	1.9413	1.2366e−02	1.9445

Table 7
The absolute value errors and the convergence rates for $\lambda_{\frac{3}{2}}$

	$\rho = 1.00$		$\rho = 0.50$	
	$\lvert\lambda_{\frac{3}{2}} - \lambda_{\frac{3}{2},h}\rvert$	Rate	$\lvert\lambda_{\frac{3}{2}} - \lambda_{\frac{3}{2},h}\rvert$	Rate
$h = \frac{1}{8}$	1.5011e−01		1.5399e−01	
$h = \frac{1}{16}$	3.5198e−02	2.0924	3.5461e−02	2.1185
$h = \frac{1}{32}$	9.0060e−03	1.9665	9.0973e−03	1.9882
$h = \frac{1}{64}$	2.2780e−03	1.9831	2.2722e−03	2.0013

where

$$f_1 = \begin{cases} -2(y-1)y^2 - (x^2-1)(6y-2) & \text{if } y \geqslant 0, \\ -2(y+1)y^2 - (x+1)(6xy+2x+12y^2+6y) & \text{if } y < 0. \end{cases}$$

The exact solution of the underlying problem is then

$$u = w + \eta_\rho s_{\frac{1}{2}} + \eta_\rho s_{\frac{3}{2}},$$

where the regular part w is given by

$$w = (\eta_2 - \eta_\rho)(s_{\frac{1}{2}} + s_{\frac{3}{2}}) + \begin{cases} (x^2-1)(y-1)y^2 & \text{if } y \geqslant 0, \\ (x+1)(y+1)(x+y)y^2 & \text{if } y \leqslant 0. \end{cases}$$

Note that the function w is in $H^2(\Omega)$, but not in $H^3(\Omega)$. Numerical results for the discretization accuracy of the finite element approximation to w are given in Tables 4 and 5. Results for the stress intensity factors are contained in Tables 6 and 7.

References

[1] I. Babuska, Finite element method for domains with corners, Computing 6 (1970) 264–273.
[2] I. Babuska, R.B. Kellogg, J. Pitkäranta, Direct and inverse error estimates for finite elements with mesh refinements, Numer. Math. 33 (1979) 447–471.
[3] I. Babuska, A. Miller, The post-processing approach in the finite element method—Part 2: The calculation of stress intensity factors, Int. J. Numer. Methods Engrg. 20 (1984) 1111–1129.
[4] I. Babuska, M. Suri, The *h–p* version of the finite element method with quasiuniform meshes, RAIRO Modél. Math. Anal. Numér. 21 (1987) 199–238.
[5] H. Blum, M. Dobrowolski, On finite element methods for elliptic equations on domains with corners, Computing 28 (1982) 53–63.
[6] H. Blum, Numerical treatment of corner and crack singularities, Dissertation, Bonn, 1981.
[7] M. Bourlard, M. Dauge, M.-S. Lubuma, S. Nicaise, Coefficients of the singularities for elliptic boundary value problems on domains with conical points III. Finite element methods on polygonal domains, SIAM Numer. Anal. 29 (1992) 136–155.
[8] S. Brenner, Multigrid methods for the computation of singular solutions and stress intensity factor I: Corner singularities, Math. Comput. 68 (226) (1999) 559–583.
[9] Z. Cai, S. Kim, A finite element method using singular functions for the Poisson equation: corner singularities, SIAM J. Numer. Anal. 39 (2001) 286–299.
[10] Z. Cai, S. Kim, B. Shin, Solution methods for the Poisson equation: corner singularities, SIAM J. Sci. Comput. 39 (2001) 286–299.
[11] Z. Cai, S. Kim, G. Woo, A finite element method using singular functions for the Poisson equation: crack singularities, Numer. Linear Algebra Appl. 9 (2002) 445–455.
[12] M. Dauge, Elliptic boundary value problems on corner domainsLecture Notes in Mathematics, vol. 1341, Springer-Verlag, Berlin–Heidelberg, 1988.
[13] P. Destuynder, M. Djaoua, Estimation de l'erreur sur le coefficient de la singularité de la solution d'un probléme elliptique sur un ouvert avec coin, RAIRO Ser. Rouge 14 (1980) 239–248.
[14] M. Djaoua, Equations Intégrales pour un Probleme Singulier dans le Plan, These de Troisieme Cycle, Universite Pierre et Marie Curie, Paris, 1977.
[15] M. Dobrowolski, Numerical Approximation of Elliptic Interface and Corner Problems, Habilitation-schrift, Bonn, 1981.
[16] G.J. Fix, S. Gulati, G.I. Wakoff, On the use of singular functions with finite elements approximations, J. Comput. Phys. 13 (1973) 209–228.
[17] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman, Boston, MA, 1985.
[18] U. Rüde, Mathematical and computational techniques for multilevel adaptive methodsFrontiers in Applied Mathematics, vol. 13, SIAM, Philadelphia, 1993.
[19] A. Schatz, L. Wahlbin, Maximum norm estimates in the finite element method on plane polygonal domains, Part 1, Math. Comput. 32 (141) (1978) 73–109;
A. Schatz, L. Wahlbin, Maximum norm estimates in the finite element method on plane polygonal domains, Part 2 (refinements), Math. Comput. 33 (146) (1979) 465–492.
[20] H. Yserentant, The convergence of multi-level methods for solving finite element equations in the presence of singularities, Math. Comput. 47 (1986) 399–409.

[21] S. Zhang, Optimal-order nonnested multigrid methods for solving finite element equations. I: On quasi-uniform meshes, Math. Comput. 55 (1990) 23–36;
S. Zhang, Optimal-order nonnested multigrid methods for solving finite element equations. II: On non-quasi-uniform meshes, Math. Comput. 55 (1990) 439–450;
S. Zhang, Optimal-order nonnested multigrid methods for solving finite element equations. III: On degenerate meshes, Math. Comput. 64 (1995) 23–49.