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RECOVERY-BASED ERROR ESTIMATORS FOR INTERFACE PROBLEMS: MIXED AND NONCONFORMING FINITE ELEMENTS*

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Abstract. In [Z. Cai and S. Zhang, *SIAM J. Numer. Anal.*, 47 (2009), pp. 2132–2156], we introduced and analyzed a recovery-based a posteriori error estimator for conforming linear finite element approximation to interface problems. It was shown theoretically that the estimator is robust with respect to the size of jumps provided that the distribution of coefficients is locally monotone. Numerical examples showed that this condition is unnecessary. This paper extends the idea in [Z. Cai and S. Zhang, *SIAM J. Numer. Anal.*, 47 (2009), pp. 2132–2156] to mixed and nonconforming finite element methods for developing and analyzing robust estimators. Numerical results on test problems are also presented. Moreover, an a priori error estimate is obtained when the underlying problem has low regularity.

Key words. a posteriori error estimator, mixed element, nonconforming element, interface problems

AMS subject classifications. 65N30, 65N15

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1. Introduction.

The recovery-based a posteriori error estimators have been extensively studied for conforming finite elements by many researchers due to their many appealing properties: simplicity, asymptotic exactness, and universality. For the mixed and nonconforming finite element methods, Carstensen and Bartels in [11] introduced and analyzed recovery-based error estimators. Their estimators for both the mixed and the nonconforming elements are based on the recovery of the gradient in $H^1(\Omega)^2$. These estimators work well for the Poisson equation even though the gradient of the exact solution belongs only to $H(\text{div}) \cap H(\text{curl})$ for nonconvex polygonal domains. For other types of estimators on the mixed and the nonconforming methods, see [1, 2, 10, 11, 12, 14, 16, 17, 26, 27] and references therein.

As demonstrated numerically in [23, 9] and theoretically in [9], for conforming finite element approximations to the interface problem with large jumps, existing estimators of the recovery type overrefine regions where there are no errors and, hence, fail to reduce the global error. This is also true for the recovery-based estimators in [11] for the mixed and nonconforming finite element methods (see Figures 1, 2, 7, and 8). The reason for the overrefinements is that the recovered gradient is continuous but the true gradient is discontinuous. To overcome this structural difficulty, one often applies the method on each subdomain separately. For reasons why this local approach is not favorable, see detailed discussions in [23]. More importantly, the local approach fails when triangulations do not align with interfaces, which occurs when interfaces are curves/surfaces or have unknown locations. In [9], we introduced and analyzed a global approach for the conforming linear finite element approximation by recovering the flux in the $H(\text{div})$ conforming finite element spaces. The resulting

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estimator is then free of overrefinements and satisfies the efficiency and reliability bounds with constants independent of the size of jumps.

The purpose of this paper is to extend the idea in [9] to mixed and nonconforming finite element approximations. To do so, we need to determine what quantities are to be recovered and which finite element spaces are to be used. The guideline for such choices is based on our view that a recovery-based estimator is a measurement of the violation of finite element approximations on physical continuities. Therefore, the quantities to be recovered are those whose finite element approximations do not preserve the physical continuity. The interface problems in (2.1) have two physical continuities: the solution u and the normal component of the flux $\boldsymbol{\sigma} = -k\nabla u$. Mathematically, this means

$$u \in H^1(\Omega) \quad \text{and} \quad \boldsymbol{\sigma} \in H(\text{div}) \subset L^2(\Omega)^2. \tag{1.1}$$

For the mixed method, the continuity of the solution is violated while that of the flux is preserved. To measure such a violation, we recover the gradient of the solution. To choose proper finite element spaces, we notice that the first property in (1.1) implies

$$\nabla u \in H(\text{curl}). \tag{1.2}$$

Physically, the tangential components of vector fields in $H(\text{curl})$ are continuous. Therefore, the quantity to be recovered is the gradient and the proper finite element space is the $H(\text{curl})$ conforming finite element space. This choice accommodates discontinuity of the normal component of the gradient and, hence, eliminates overrefinements. For nonconforming finite element approximations, since both continuities are violated, we recover both the flux and the gradient in the $H(\text{div})$ and $H(\text{curl})$ conforming finite element spaces, respectively, through weighted L^2 projections. The estimator is then the average of two measurements: the weighted L^2 norms of differences between the direct and the recovered approximations of the flux and the gradient.

Estimators introduced in this paper are analyzed by establishing the reliability and efficiency bounds and are supported by numerical results. In particular, we prove theoretically that the estimators are robust, in the sense that the reliability and efficiency constants are independent of the size of jumps, provided that the distribution of coefficients is locally monotone. We also show numerically that there is no overrefinements along interfaces for a benchmark test problem whose coefficients are not locally monotone. Results in this paper may be extended to three dimensions in a straightforward manner.

It is important to point out that research on robust estimators for interface problems is limited. For the conforming finite element method, robust a posteriori error estimators have been studied by Bernardi and Verfürth [6] and Petzoldt [24] for the residual-based estimator, by Luce and Wohlmuth [18] for the equilibrated estimator, and by us [9] for the recovery-based estimator. For the nonconforming elements, Ainsworth [1] studied a robust equilibrated estimator. For the mixed method, see a recent work by Ainsworth [2].

This paper is organized as follows. Section 2 introduces interface problems and variational formulations. Various finite element spaces and both mixed and nonconforming finite element approximations are described in section 3. Recovery procedures and a posteriori error estimators are defined in sections 4 and 5, respectively. We establish the efficiency and reliability bounds of estimators introduced in this paper in section 6. Finally, we present numerical results for a benchmark test problem in section 7.

1.1. Function spaces and preliminaries. Let Ω be a bounded polygonal domain in $\Re^2$ with boundary $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. For a subdomain $G \subset \Omega$, we use the standard notations and definitions for the Sobolev spaces $H^s(G)$ and $H^s(\partial G)$ for $s \geq 0$. The standard associated inner products are denoted by $(\cdot,\, \cdot)_{s,G}$ and $(\cdot,\, \cdot)_{s,\partial G}$, and their respective norms are denoted by $\|\cdot\|_{s,G}$ and $\|\cdot\|_{s,\partial G}$. We omit the subscript G or ∂G if $G = \Omega$ from the inner product and norm designation when there is no risk of confusion. We shall use the following subspaces of $H^1(\Omega)$:

$$H^1_D(\Omega) := \{v \in H^1(\Omega) \,:\, v = 0 \text{ on } \Gamma_D\} \quad \text{and} \quad H^1_N(\Omega) := \{v \in H^1(\Omega) \,:\, v = 0 \text{ on } \Gamma_N\}.$$

In two dimensions, for $\boldsymbol{\tau} = (\tau_1,\, \tau_2)^t$, define the divergence and curl operators by

$$\nabla \cdot \boldsymbol{\tau} := \frac{\partial \tau_1}{\partial x_1} + \frac{\partial \tau_2}{\partial x_2} \quad \text{and} \quad \nabla\times\boldsymbol{\tau} := \frac{\partial \tau_2}{\partial x_1} - \frac{\partial \tau_1}{\partial x_2},$$

respectively. For a scalar-valued function v, define the operator $\nabla^\perp$ by

$$\nabla^\perp v = Q\,\nabla v = \left(-\frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_1}\right)^t \quad \text{with} \quad Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We shall use the following Hilbert spaces:

$$H(\mathrm{div};\Omega) = \{\boldsymbol{\tau} \in L^2(\Omega)^2 \,:\, \nabla\cdot\boldsymbol{\tau} \in L^2(\Omega)\}$$

and

$$H(\mathrm{curl};\Omega) = \{\boldsymbol{\tau} \in L^2(\Omega)^2 \,:\, \nabla\times\boldsymbol{\tau} \in L^2(\Omega)\},$$

equipped with the norms

$$\|\boldsymbol{\tau}\|_{H(\mathrm{div};\,\Omega)} = \left(\|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\nabla\cdot\boldsymbol{\tau}\|^2_{0,\Omega}\right)^{\frac{1}{2}}$$

and

$$\|\boldsymbol{\tau}\|_{H(\mathrm{curl};\,\Omega)} = \left(\|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\nabla\times\boldsymbol{\tau}\|^2_{0,\Omega}\right)^{\frac{1}{2}},$$

respectively. Denote their subspaces by

$$H_N(\mathrm{div};\Omega) = \{\boldsymbol{\tau} \in H(\mathrm{div};\Omega) : \boldsymbol{\tau}\cdot\mathbf{n}\big|_{\Gamma_N} = 0\}$$

and

$$H_D(\mathrm{curl};\Omega) = \{\boldsymbol{\tau} \in H(\mathrm{curl};\Omega) : \boldsymbol{\tau}\cdot\mathbf{t}\big|_{\Gamma_D} = 0\},$$

where $\mathbf{n} = (n_1, n_2)^t$ and $\mathbf{t} = (t_1, t_2)^t = Q\,\mathbf{n} = (-n_2, n_1)^t$ are the unit vectors outward normal to and clockwise tangent to the boundary $\partial\Omega$, respectively.

2. Interface problems and variational forms. Consider the following interface problem:

$$
\begin{cases} -\nabla\cdot\,(k(x)\nabla\, u) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n}\cdot(k\nabla\, u) = 0 \quad \text{on } \Gamma_N, \end{cases} \tag{2.1}
$$

where f is a given scalar-valued function in $L^2(\Omega)$, and $k(x)$ is positive and piecewise constant on polygonal subdomains of Ω with possible large jumps across subdomain boundaries (interfaces): $k(x) = k_i > 0$ in Ω_i for $i = 1, \ldots, n$. Here, $\{\Omega_i\}_{i=1}^n$ is a partition of the domain Ω with Ω_i being an open polygonal domain. For simplicity, we consider only homogeneous boundary conditions. Also, we assume that Γ_D is not empty (i.e., $\text{mes}\,(\Gamma_D) \neq 0$).

Denote bilinear and linear forms, respectively, by

$$
a(u,\, v) = (k(x)\nabla u,\, \nabla v) \quad \text{and} \quad f(v) = (f,\, v).
$$

Then the variational form of problem (2.1) is to find $u \in H^1_D(\Omega)$ such that

$$
a(u,\, v) = f(v) \quad \forall\, v \in H^1_D(\Omega). \tag{2.2}
$$

Define the flux by

$$
\boldsymbol{\sigma} = -k(x)\nabla u \quad \text{in } \Omega,
$$

and then the mixed variational formulation is to find $(\boldsymbol{\sigma},\, u) \in H_N(\text{div};\Omega) \times L^2(\Omega)$ such that

$$
\begin{cases} (k^{-1}\boldsymbol{\sigma},\, \boldsymbol{\tau}) - (\nabla\cdot\boldsymbol{\tau},\, u) = 0 & \forall\, \boldsymbol{\tau} \in H_N(\text{div};\Omega), \\ (\nabla\cdot\boldsymbol{\sigma},\, v) = (f,\, v) & \forall\, v \in L^2(\Omega). \end{cases} \tag{2.3}
$$

3. Finite element approximations.

3.1. Finite element spaces. For simplicity of presentation, consider only triangular elements. Let $\mathcal{T} = \{K\}$ be a finite element partition of the domain Ω and denote by h_K the diameter of element K. Assume that the triangulation $\mathcal{T}$ is regular [13] and that interfaces $F = \{\partial\Omega_i \cap \partial\Omega_j \,|\, i, j = 1, ..., n\}$ do not cut through any element $K \in \mathcal{T}$.

Denote the set of all edges of the triangulation by $\mathcal{E} := \mathcal{E}_\Omega \cup \mathcal{E}_D \cup \mathcal{E}_N$, where $\mathcal{E}_\Omega$ is the set of all interior element edges and $\mathcal{E}_D$ and $\mathcal{E}_N$ are the sets of all boundary edges belonging to the respective Γ_D and Γ_N. For each $e \in \mathcal{E}$, denote by m_e and h_e the midpoint and the length of the edge e, respectively. For each $K \in \mathcal{T}$, let $P_k(K)$ be the space of polynomials of degree k. Denote the conforming and nonconforming linear finite element spaces [13, 15] associated with the triangulation $\mathcal{T}$ by

$$
\begin{aligned} &\mathcal{U} = \{v \in H^1(\Omega) \,:\, v|_K \in P_1(K) \quad \forall\, K \in \mathcal{T}\} \\ \textit{and} \quad &\mathcal{U}^{nc} = \{v \in L^2(\Omega) \,:\, v|_K \in P_1(K)\; \forall\, K \in \mathcal{T},\ \text{and } v \text{ is continuous at } m_e\; \forall\, e \in \mathcal{E}_\Omega\}, \end{aligned}
$$

respectively, and their respective subspaces by

$$
\mathcal{U}_D = \{v \in \mathcal{U} :\, v = 0 \text{ on } \Gamma_D\} \quad \text{and} \quad \mathcal{U}^{nc}_D = \{v \in \mathcal{U}^{nc} \,:\, v(m_e) = 0\; \forall\, e \in \mathcal{E}_D\}.
$$

The $H(\text{div};\,\Omega)$ conforming Raviart–Thomas (RT) and Brezzi–Douglas–Marini (BDM) spaces [7] of the lowest order are defined by

$$
RT_0 = \{\boldsymbol{\tau} \in H_N(\text{div};\,\Omega) : \boldsymbol{\tau}|_K \in RT_0(K)\; \forall\, K \in \mathcal{T}\} \text{ with } RT_0(K) = P_0(K)^2 + (x_1, x_2)\, P_0(K)
$$

and

$$BDM_1 = \{\boldsymbol{\tau} \in H_N(\text{div};\Omega) : \boldsymbol{\tau}|_K \in BDM_1(K) \ \ \forall\, K \in \mathcal{T}\} \text{ with } BDM_1(K) = P_1(K)^2,$$

respectively. The $H(\text{curl}\,;\Omega)$ conforming first and second types of Nedelec spaces of the lowest order [21, 22] are defined by

$$ND_1 = \{\boldsymbol{\tau} \in H_D(\text{curl};\Omega) : \boldsymbol{\tau}|_K \in ND_1(K)\,\forall\, K \in \mathcal{T}\} \text{ with } ND_1(K) = P_0(K)^2 + (x_2, -x_1)P_0(K)$$

and

$$ND_2 = \{\boldsymbol{\tau} \in H_D(\text{curl};\Omega) : \boldsymbol{\tau}|_K \in ND_2(K)\ \forall\, K \in \mathcal{T}\} \text{ with } ND_2(K) = P_1(K)^2,$$

respectively. For convenience, denote RT_0/BDM_1 by $\mathcal{V}_N$ and ND_1/ND_2 by $\mathcal{W}_D$. Also, let

$$P_0 = \{v \in L^2(\Omega) : v|_K \in P_0(K)\ \forall\, K \in \mathcal{T}\}.$$

Finally, we define the discrete gradient, divergence, and curl operators as follows:

$$(\nabla_h v)|_K := \nabla(v|_K), \quad (\nabla_h \cdot \boldsymbol{\tau})|_K := \nabla \cdot (\boldsymbol{\tau}|_K), \quad \text{and} \quad (\nabla_h \times \boldsymbol{\tau})|_K := \nabla\times(\boldsymbol{\tau}|_K)$$

for all $K \in \mathcal{T}$, respectively.

3.2. Finite element approximations. The mixed finite element method is to find $(\boldsymbol{\sigma}_m, u_m) \in \mathcal{V}_N \times P_0$ such that

$$\text{(3.1)} \qquad \begin{cases} (k^{-1}\boldsymbol{\sigma}_m,\, \boldsymbol{\tau}) - (\nabla\cdot\boldsymbol{\tau},\, u_m) \,=\, 0 & \forall\, \boldsymbol{\tau} \in \mathcal{V}_N, \\ \qquad\qquad (\nabla\cdot\boldsymbol{\sigma}_m,\, v) \,=\, (f,\, v) & \forall\, v \in P_0. \end{cases}$$

Let $(\boldsymbol{\sigma},\, u)$ and $(\boldsymbol{\sigma}_m, u_m)$ be the solutions of (2.3) and (3.1), respectively, and denote the true error of the mixed finite element approximation by

$$\text{(3.2)} \qquad (E_m,\, e_m) = (\boldsymbol{\sigma} - \boldsymbol{\sigma}_m,\, u - u_m).$$

Then the difference between (2.3) and (3.1) gives the following error equations:

$$\text{(3.3)} \qquad \begin{cases} (k^{-1}E_m,\, \boldsymbol{\tau}) - (\nabla\cdot\boldsymbol{\tau},\, e_m) \,=\, 0 & \forall\, \boldsymbol{\tau} \in \mathcal{V}_N, \\ \qquad\qquad (\nabla\cdot E_m,\, v) \,=\, 0 & \forall\, v \in P_0. \end{cases}$$

A standard argument gives the following a priori error estimate.

THEOREM 3.1. *Assume that the solution, $(\boldsymbol{\sigma}, u)$, of problem (2.3) belongs to $H^s(\Omega) \times H^{1+s}(\Omega)$ with $0 \le s \le 1$. Then we have the following a priori error bound:*

$$\text{(3.4)} \qquad \|k^{-1/2}E_m\|_{0,\Omega} \le C\,\|h^s k^{1/2}\nabla u\|_{s,\Omega}$$

with $\|h^s k^{1/2}\nabla u\|_{s,\Omega} = \left(\sum_{K\in\mathcal{T}} h_K^{2s}\,\|k^{1/2}\nabla u\|^2_{s,K}\right)^{1/2}$. Here and hereafter, in this paper, we use C with or without subscripts to denote a generic positive constant, possibly different at different occurrences, that is independent of the mesh parameter h_K and the ratio $k_{\max}/k_{\min}$ but may depend on the domain Ω.

The nonconforming finite element method is to find $u_{nc} \in \mathcal{U}_D^{nc}$ such that

$$\text{(3.5)} \qquad (k\nabla_h u_{nc},\, \nabla_h v) = (f, v) \qquad \forall\, v \in \mathcal{U}_D^{nc}.$$

Let u and u_{nc} be the solutions of (2.2) and (3.5), respectively, and denote the true error of the nonconforming finite element approximation by $e_{nc} = u - u_{nc}$. Since $\mathcal{U}_D \subset \mathcal{U}_D^{nc}$, we have the following error equation:

$$(k\nabla_h e_{nc},\, \nabla v) = (k(\nabla u - \nabla_h u_{nc}),\, \nabla v) = 0 \quad \forall\ v \in \mathcal{U}_D. \tag{3.6}$$

For $K \in \mathcal{T}$ and any $v \in H^{1+s}(K)$, $0 < s \le 1$, denote by

$$B_{s,K}(h_K,\, v) = \begin{cases} h_K^s \|k^{1/2}\nabla v\|_{s,K}, & s \in (1/2,\, 1], \\ h_K^s \|k^{1/2}\nabla v\|_{s,K} + h_K k_K^{-1/2}\|f\|_{0,K}, & s \in (0,\, 1/2]. \end{cases}$$

LEMMA 3.2. *For any $w \in \mathcal{U}_D^{nc}$ and any $K \in \mathcal{T}$, let $\bar{w}_{K,e}$ be the mean value of $w|_K$ over edge $e \in \partial K$. Assume that the solution u of problem* (2.2) *belongs to $H^{1+s}(\Omega)$ with $s \in (0,\, 1]$, and then*

$$\sum_{e\in\partial K} \left| \int_e (\mathbf{n}\cdot k\nabla u)\,(w - \bar{w}_{K,e})\,ds \right| \le C\, B_{s,K}(h_K,\, u)\, \|\nabla w\|_{0,K}. \tag{3.7}$$

For $v \in H^{1+s}(K)$, note that integral $\int_e (\mathbf{n}\cdot k\nabla v)\, w ds$ is the standard integration in $L^2(e)$ if $s > 1/2$. When $s \in (0, 1/2]$, it should be viewed as the duality pairing $\langle(\mathbf{n}\cdot k\nabla v), w\rangle_e$, where $(\mathbf{n}\cdot k\nabla v)|_e \in H^{\epsilon-1/2}(e)$ and $w|_e \in H^{1/2-\epsilon}(e)$ for any positive $\epsilon < s$.

Proof. The definition of $\bar{w}_{K,e}$ implies

$$\int_e (\mathbf{n}\cdot k\nabla u)\,(w - \bar{w}_{K,e})\,ds = \int_e (\mathbf{n}\cdot k\nabla u - \zeta_e)\,(w - \bar{w}_{K,e})\,ds, \tag{3.8}$$

for any constant ζ_e. When $s \in (1/2,\, 1]$, (3.7) may be proved by a standard argument (see, e.g., [8]). When $s \in (0,\, 1/2]$, using (3.8), the definition of the dual norm, and the approximation property, we have

$$\begin{aligned} \left| \int_e (\mathbf{n}\cdot k\nabla u)\,(w - \bar{w}_{K,e})\,ds \right| &\le \|\mathbf{n}\cdot k\nabla u - \zeta_e\|_{-1/2+\epsilon,e}\|w - \bar{w}_{K,e}\|_{1/2-\epsilon,e} \\ &\le C\, h^\epsilon \|\mathbf{n}\cdot k\nabla u - \zeta_e\|_{-1/2+\epsilon,e}\|\nabla w\|_{0,K}. \end{aligned}$$

Let $\bar{\boldsymbol{\zeta}}$ be the mean value of $k\nabla u$ over K. Choosing $\zeta_e = \bar{\boldsymbol{\zeta}}\cdot\mathbf{n}_e$ and using (2.1) and the fact [5] that $\|\nabla\phi\cdot\mathbf{n}\|_{-1/2+\epsilon,e} \le C\left(\|\nabla\phi\|_{\epsilon,K} + h_K^{1-\epsilon}\|\Delta\phi\|_{0,K}\right)$ for any $\phi \in H^{1+\epsilon}(K)$ with $\Delta\phi \in L^2(K)$ and for any $0 < \epsilon < 1/2$, we have

$$\begin{aligned} &\|\mathbf{n}\cdot k\nabla u - \zeta_e\|_{-1/2+\epsilon,e} = \|\mathbf{n}\cdot(k\nabla u - \boldsymbol{\zeta})\|_{-1/2+\epsilon,e} \\ &\le C\left(\|k\nabla u - \bar{\boldsymbol{\zeta}}\|_{\epsilon,K} + h_K^{1-\epsilon}\|f\|_{0,K}\right) \le C\left(h_K^{s-\epsilon}\|k\nabla u\|_{s,K} + h_K^{1-\epsilon}\|f\|_{0,K}\right). \end{aligned}$$

Combining the above two inequalities yields (3.7) and, hence, the proof of the lemma. ∎

THEOREM 3.3. *Assume that the solution u of problem* (2.2) *belongs to $H^{1+s}(\Omega)$ with $0 < s \le 1$. Then we have the following a priori error bound:*

$$\|k^{1/2}\nabla_h(u - u_{nc})\|_{0,\Omega} \le C\left(\sum_{K\in\mathcal{T}} B_{s,K}^2(h_K,\, u)\right)^{1/2}.$$

Proof. Let u_c be the orthogonal projection of u onto $\mathcal{U}_D$ with respect to the inner product $(k\nabla\cdot, \nabla\cdot)$, and then the fact that $\mathcal{U}_D \subset \mathcal{U}_D^{nc}$ implies

$$\inf_{v\in\mathcal{U}_D^{nc}} \|k^{1/2}\nabla_h(u-v)\|_{0,\Omega} \le \|k^{1/2}\nabla_h(u-u_c)\|_{0,\Omega} \le C\left(\sum_{K\in\mathcal{T}} h_K^{2s}\|k^{1/2}\nabla u\|_{s,K}^2\right)^{1/2}.$$

The second inequality above may be proved similarly as that of Proposition 2.4 in [6].

For any $w \in \mathcal{U}_D^{nc}$, any $K \in \mathcal{T}$, and any $e \in \partial K$, let K^- be the element sharing the common edge e, and then the continuity of w at the midpoint of e implies that $\bar{w}_{K,e} = \bar{w}_{K^-,e}$. For $e \in \mathcal{E}_D$, $w(m_e) = 0$ implies that $\bar{w}_{K,e} = 0$. Then it follows from integration by parts, (2.1), and the continuity of $\mathbf{n}\cdot k\nabla u$ across edge e that

$$(k\nabla u,\, \nabla_h w) - (f,\, w) = \sum_{K\in\mathcal{T}}\int_{\partial K}(\mathbf{n}\cdot k\nabla u)\, w\, ds = \sum_{K\in\mathcal{T}}\sum_{e\in\partial K}\int_e(\mathbf{n}\cdot k\nabla u)\,(w-\bar{w}_{K,e})\, ds,$$

which, together with the triangle inequality and Lemma 3.2, give

$$|(k\nabla u,\, \nabla_h w) - (f,\, w)| \le C\left(\sum_{K\in\mathcal{T}} B_{s,K}^2(h_K,\, u)\right)^{1/2} \|k^{1/2}\nabla_h w\|_{0,\Omega}.$$

Now, Theorem 3.3 is a direct consequence of Strang's lemma (see, e.g., [13]). ∎

4. Gradient and/or flux recovery.

4.1. Implicit approximation. For the mixed finite element approximation $(\boldsymbol{\sigma}_m,\, u_m)$, the continuity of the solution and, hence, the continuity of the tangential component of the gradient are violated while that of the flux is preserved. This suggests to recover the gradient in the $H(\text{curl})$ conforming finite element space ND_2. Since $\nabla u = -k^{-1}\boldsymbol{\sigma}$, it is recovered by finding $\boldsymbol{\rho}_m \in ND_2$ such that

$$(k\,\boldsymbol{\rho}_m,\, \boldsymbol{\tau}) = -(\boldsymbol{\sigma}_m,\, \boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in ND_2. \tag{4.1}$$

It may be proved easily that $\boldsymbol{\rho}_m$ is a good approximation to ∇u up to the accuracy of the mixed finite element approximation.

THEOREM 4.1 (see [8]). *There exists a constant $C > 0$ independent of the ratio $k_{\max}/k_{\min}$ such that*

$$\|k^{1/2}(\nabla u - \boldsymbol{\rho}_m)\|_{0,\Omega} \le C\left(\inf_{\boldsymbol{\tau}\in ND_2}\|k^{1/2}(\nabla u - \boldsymbol{\tau})\|_{0,\Omega} + \|k^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_m)\|_{0,\Omega}\right). \tag{4.2}$$

For the nonconforming finite element approximation u_{nc}, both the continuities of the tangential component of the gradient and the normal component of the flux are violated. Hence, we recover both the gradient and flux as follows: finding $\boldsymbol{\rho}_{nc} \in \mathcal{W}_D$ such that

$$(k\boldsymbol{\rho}_{nc},\, \boldsymbol{\tau}) = (k\nabla_h u_{nc},\, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathcal{W}_D, \tag{4.3}$$

and finding $\boldsymbol{\sigma}_{nc} \in \mathcal{V}_N$ such that

$$(k^{-1}\boldsymbol{\sigma}_{nc},\, \boldsymbol{\tau}) = -(\nabla_h u_{nc},\, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathcal{V}_N. \tag{4.4}$$

THEOREM 4.2 (see [8]). *There exists a constant $C > 0$ independent of the ratio* $k_{\max}/k_{\min}$

$$\|k^{1/2}(\nabla u - \boldsymbol{\rho}_{nc})\|_{0,\Omega} \le C\left(\inf_{\boldsymbol{\tau}\in\mathcal{W}_D} \|k^{1/2}(\nabla u - \boldsymbol{\tau})\|_{0,\Omega} + \|k^{1/2}(\nabla u - \nabla_h u_{nc})\|_{0,\Omega}\right) \tag{4.5}$$

and that

$$\|k^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{nc})\|_{0,\Omega} \le C\left(\inf_{\boldsymbol{\tau}\in\mathcal{V}_N} \|k^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\tau})\|_{0,\Omega} + \|k^{1/2}(\nabla u - \nabla_h u_{nc})\|_{0,\Omega}\right). \tag{4.6}$$

4.2. Explicit approximations. Let $\delta_{e\,e'}$ and δ_{ij} denote the Kronecker delta. Nodal basis functions of RT_0, BDM_1, ND_1, and ND_2 corresponding to edge $e \in \mathcal{E}$ are characterized as follows:

(1) For RT_0, $\boldsymbol{\phi}_e$ is uniquely determined by

$$\int_{e'} \boldsymbol{\phi}_e \cdot \mathbf{n}_{e'}\, ds = \delta_{e\,e'} \quad \forall\ e' \in \mathcal{E}.$$

(2) For BDM_1, $\boldsymbol{\phi}_{e,i}$ $(i = 1, 2)$ are uniquely determined by

$$\int_{e'} s^{j-1} \boldsymbol{\phi}_{e,i} \cdot \mathbf{n}_{e'}\, ds = \delta_{e\,e'}\delta_{ij} \ \ \forall\ e' \in \mathcal{E} \ \text{ and for } \ j = 1,\, 2,$$

where s is a local coordinate on e' ranging from -1 to 1. Notice that $\boldsymbol{\phi}_{e,1}$ of BDM_1 and $\boldsymbol{\phi}$ of RT_0 are the same. Since $\boldsymbol{\phi}_{e,1}|_e \cdot \mathbf{n}_e = 1/|e|$, we have the following orthogonality property:

$$\int_e (\boldsymbol{\phi}_{e,1} \cdot \mathbf{n}_e)(\boldsymbol{\phi}_{e,2} \cdot \mathbf{n}_e) ds = \frac{1}{|e|} \int_e (\boldsymbol{\phi}_{e,2} \cdot \mathbf{n}_e) ds = 0.$$

(3) For ND_1, $\boldsymbol{\psi}_e$ is uniquely determined by

$$\int_{e'} \boldsymbol{\psi}_e \cdot \mathbf{t}_{e'}\, ds = \delta_{e\,e'} \quad \forall\ e' \in \mathcal{E}.$$

(4) For ND_2, $\boldsymbol{\psi}_{e,i}$ $(i = 1, 2)$ are uniquely determined by

$$\int_{e'} s^{j-1} \boldsymbol{\psi}_{e,i} \cdot \mathbf{t}_{e'}\, ds = \delta_{e\,e'}\delta_{ij} \ \ \forall\ e' \in \mathcal{E} \ \text{ and for } \ j = 1,\, 2.$$

Similarly, $\boldsymbol{\psi}_{e,1}$ of ND_2 and $\boldsymbol{\psi}$ of ND_1 coincide and satisfy the following orthogonality:

$$\int_e (\boldsymbol{\psi}_{e,1} \cdot \mathbf{t}_e)(\boldsymbol{\psi}_{e,2} \cdot \mathbf{t}_e) ds = 0. \tag{4.7}$$

LEMMA 4.1. *Every constant vector $\boldsymbol{\tau}$ on $K \in \mathcal{T}$ has the following representations:*

$$\boldsymbol{\tau} = \sum_{e\in\partial K} \tau_e \boldsymbol{\phi}_e,\ \tau_e = \int_e (\boldsymbol{\tau} \cdot \mathbf{n}_e)\, ds \quad \textit{and} \quad \boldsymbol{\tau} = \sum_{e\in\partial K} \tau_e\, \boldsymbol{\psi}_e,\ \tau_e = \int_e (\boldsymbol{\tau} \cdot \mathbf{t}_e)\, ds,$$

respectively, in RT_0 and ND_1. Every linear vector $\boldsymbol{\tau}$ on K has the following representations:

$$\boldsymbol{\tau} = \sum_{e\in\partial K} (\tau_{e,1}\, \boldsymbol{\phi}_{e,1} + \tau_{e,2}\, \boldsymbol{\phi}_{e,2}) \quad \textit{with} \quad \tau_{e,i} = \int_e s^{i-1} (\boldsymbol{\tau} \cdot \mathbf{n}_e)\, ds$$

and

$$\boldsymbol{\tau} = \sum_{e\in\partial K} (\tau_{e,1}\,\boldsymbol{\psi}_{e,1} + \tau_{e,2}\,\boldsymbol{\psi}_{e,2}) \quad \text{with} \quad \tau_{e,i} = \int_e s^{i-1}(\boldsymbol{\tau}\cdot\mathbf{t}_e)\,ds,$$

respectively, in BDM_1 *and* ND_2.

Proof. Since $RT_0(K)$ and $ND_1(K)$ and both $BDM_1(K)$ and $ND_2(K)$ contain the respective constant and linear vectors, the lemma is then a direct consequence of the characteristic equations of nodal basis functions. □

LEMMA 4.2. *For a linear function* v *defined on edge* e*, let* $\{\boldsymbol{\psi}_{e,i}\}_{i=1}^2$ *be the* ND_2 *basis functions, and then*

$$v = v_1\,\boldsymbol{\psi}_{e,1}\cdot\mathbf{t}_e + v_2\,\boldsymbol{\psi}_{e,2}\cdot\mathbf{t}_e \quad \text{with} \quad v_i = \int_e s^{i-1} v\,ds \tag{4.8}$$

and

$$\int_e |v|^2\,ds = v_1^2 \int_e |\boldsymbol{\psi}_{e,1}|^2 ds + v_2^2 \int_e |\boldsymbol{\psi}_{e,2}|^2\,ds. \tag{4.9}$$

Proof. (4.8) follows from the fact that $\text{span}\{\boldsymbol{\psi}_{e,i}\cdot\mathbf{t}_e\}_{i=1}^2 = \text{span}\{1,\, s\}$. (4.9) is a consequence of (4.8) and (4.7). □

Now we are ready to introduce explicit approximations to the flux and the gradient. For each $e\in\mathcal{E}$, denote by $\mathbf{n}_e$ a unit vector normal to e. When $e\in\mathcal{E}_D\cup\mathcal{E}_N$, assume that $\mathbf{n}_e$ is the unit outward normal vector. For each interior edge $e\in\mathcal{E}_\Omega$, let K_e^+ and K_e^- be the two elements sharing the common edge e such that the unit outward normal vector of K_e^+ coincides with $\mathbf{n}_e$.

For $\boldsymbol{\sigma}_m\in\mathcal{V}_N$, let $\boldsymbol{\tau}_1 = -k^{-1}\boldsymbol{\sigma}_m$. Since $\boldsymbol{\tau}_1$ is a linear vector-valued function on each K, its approximation $\hat{\boldsymbol{\rho}}_m \in ND_2$ may be defined by

$$\hat{\boldsymbol{\rho}}_m(\boldsymbol{\sigma}_m) = \sum_{e\in\mathcal{E}} \left(\hat\rho_{e,1}\boldsymbol{\psi}_{e,1} + \hat\rho_{e,2}\boldsymbol{\psi}_{e,2}\right), \tag{4.10}$$

where $\hat\rho_{e,1}$ and $\hat\rho_{e,2}$ are the zero- and one-moments of the tangential component of $\hat{\boldsymbol{\rho}}_m$ on the edge $e\in\mathcal{E}_\Omega\cup\mathcal{E}_D$, respectively, defined by

$$\hat\rho_{e,i} := \begin{cases} \gamma_{1,e}\int_e s^{i-1}(\boldsymbol{\tau}_1|_{K_e^+}\cdot\mathbf{t}_e)\,ds + (1-\gamma_{1,e})\int_e s^{i-1}(\boldsymbol{\tau}_1|_{K_e^-}\cdot\mathbf{t}_e)\,ds, & \text{for } e\in\mathcal{E}_\Omega,\\ \int_e s^{i-1}(\boldsymbol{\tau}_1\cdot\mathbf{t}_e)\,ds, & \text{for } e\in\mathcal{E}_D, \end{cases} \tag{4.11}$$

for some parameter $\gamma_{1,e}\in[0,\,1]$.

Let $\boldsymbol{\tau}_2 = -k\nabla_h u_{nc}$. Since $\boldsymbol{\tau}_2$ is piecewise constant, then it suffices to approximate it using RT_0. Define the explicit approximation $\hat{\boldsymbol{\sigma}}_{nc}(u_{nc})$ in $RT_0 = \text{span}\{\boldsymbol{\phi}_e \,:\, e\in\mathcal{E}\}$ by

$$\hat{\boldsymbol{\sigma}}_{nc}(u_{nc}) = \sum_{e\in\mathcal{E}} \hat\sigma_e\,\boldsymbol{\phi}_e, \tag{4.12}$$

where $\hat\sigma_e$ is the normal component of $\hat{\boldsymbol{\sigma}}_{nc}(u_{nc})$ on the edge $e\in\mathcal{E}$ defined by

$$\hat\sigma_e := \begin{cases} \gamma_{2,e}\int_e(\boldsymbol{\tau}_2|_{K_e^+}\cdot\mathbf{n}_e)\,ds + (1-\gamma_{2,e})\int_e(\boldsymbol{\tau}_2|_{K_e^-}\cdot\mathbf{n}_e)\,ds, & \text{for } e\in\mathcal{E}_\Omega,\\ \int_e(\boldsymbol{\tau}_2|_e\cdot\mathbf{n}_e)\,ds, & \text{for } e\in\mathcal{E}_D, \end{cases} \tag{4.13}$$

for some constant $\gamma_{2,e} \in [0, 1]$. Let $\boldsymbol{\tau}_3 = \nabla_h u_{nc}$. Since $\boldsymbol{\tau}_3$ is piecewise constant, we approximate it in $ND_1 = \text{span}\{\boldsymbol{\psi}_e : e \in \mathcal{E}\}$ by

$$\hat{\boldsymbol{\rho}}_{nc}(u_{nc}) = \sum_{e\in\mathcal{E}} \hat{\rho}_e \, \boldsymbol{\psi}_e, \tag{4.14}$$

where $\hat{\rho}_e$ is the tangential component of $\hat{\boldsymbol{\rho}}_{nc}(u_{nc})$ on the edge $e \in \mathcal{E}$ defined by

$$\hat{\rho}_e := \begin{cases} \gamma_{3,e} \int_e (\boldsymbol{\tau}_3|_{K_e^+} \cdot \mathbf{t}_e)\, ds + (1-\gamma_{3,e}) \int_e (\boldsymbol{\tau}_3|_{K_e^-} \cdot \mathbf{t}_e)\, ds, & \text{for } e \in \mathcal{E}_\Omega, \\ \int_e (\boldsymbol{\tau}_3 \cdot \mathbf{t}_e) ds, & \text{for } e \in \mathcal{E}_N, \end{cases} \tag{4.15}$$

for some constant $\gamma_{3,e} \in [0, 1]$. To ensure the efficiency bound independent of the size of jumps, we choose

$$\gamma_{2,e} = \frac{\sqrt{k_{K_e^-}}}{\sqrt{k_{K_e^+}} + \sqrt{k_{K_e^-}}} \quad \text{and} \quad \gamma_{3,e} = \frac{\sqrt{k_{K_e^+}}}{\sqrt{k_{K_e^+}} + \sqrt{k_{K_e^-}}}. \tag{4.16}$$

5. A posteriori error estimators.

5.1. Error estimators for mixed elements. Based on the solution of (4.1) and on the explicit approximation in (4.10), we define a posteriori error estimators as follows:

$$\eta_{m,K} = \|k^{1/2}(\boldsymbol{\rho}_m + k^{-1}\boldsymbol{\sigma}_m)\|_{0,K} \;\forall K \in \mathcal{T}, \qquad \eta_m = \|k^{1/2}(\boldsymbol{\rho}_m + k^{-1}\boldsymbol{\sigma}_m)\|_{0,\Omega} \tag{5.1}$$

and

$$\hat{\eta}_{m,K} = \|k^{1/2}(\hat{\boldsymbol{\rho}}_m + k^{-1}\boldsymbol{\sigma}_m)\|_{0,K} \;\forall K \in \mathcal{T}, \qquad \hat{\eta}_m = \|k^{1/2}(\hat{\boldsymbol{\rho}}_m + k^{-1}\boldsymbol{\sigma}_m)\|_{0,\Omega}, \tag{5.2}$$

respectively. These estimators essentially measure the violation of the continuity of the tangential derivatives of the true solution on the edges by the mixed elements. It is obvious that

$$\eta_m = \min_{\boldsymbol{\tau}\in ND_2} \|k^{1/2}(\boldsymbol{\tau} + k^{-1}\boldsymbol{\sigma}_m)\|_{0,\Omega} \le \hat{\eta}_m. \tag{5.3}$$

5.2. Error estimators for nonconforming elements. For every element $K \in \mathcal{T}$, let

$$\eta_{nc,1,K} = \|k^{-1/2}\boldsymbol{\sigma}_{nc} + k^{1/2}\nabla u_{nc}\|_{0,K}, \qquad \eta_{nc,2,K} = \|k^{1/2}(\boldsymbol{\rho}_{nc} - \nabla u_{nc})\|_{0,K},$$
$$\hat{\eta}_{nc,1,K} = \|k^{-1/2}\hat{\boldsymbol{\sigma}}_{nc,RT_0} + k^{1/2}\nabla u_{nc}\|_{0,K}, \quad \text{and} \quad \hat{\eta}_{nc,2,K} = \|k^{1/2}(\hat{\boldsymbol{\rho}}_{nc,N} - \nabla u_{nc})\|_{0,K},$$

where $\boldsymbol{\sigma}_{nc}$ and $\boldsymbol{\rho}_{nc}$ are the solutions of (4.4) and (4.3), respectively, and $\hat{\boldsymbol{\sigma}}_{nc,RT_0}$ and $\hat{\boldsymbol{\rho}}_{nc,N}$ are the explicit approximations defined in the respective (4.12) and (4.14). The sums of their squares are denoted by

$$\eta_{nc,1} = \|k^{-1/2}\boldsymbol{\sigma}_{nc} + k^{1/2}\nabla u_{nc}\|_{0,\Omega}, \qquad \eta_{nc,2} = \|k^{1/2}(\boldsymbol{\rho}_{nc} - \nabla u_{nc})\|_{0,\Omega},$$
$$\hat{\eta}_{nc,1} = \|k^{-1/2}\hat{\boldsymbol{\sigma}}_{nc,RT_0} + k^{1/2}\nabla u_{nc}\|_{0,\Omega}, \quad \text{and} \quad \hat{\eta}_{nc,2} = \|k^{1/2}(\hat{\boldsymbol{\rho}}_{nc,N} - \nabla u_{nc})\|_{0,\Omega}.$$

It is obvious that

$$\eta_{nc,1} = \min_{\boldsymbol{\tau}\in\mathcal{V}_N} \|k^{-1/2}\boldsymbol{\tau} + k^{1/2}\nabla_h u_{nc}\|_{0,\Omega} \quad \text{and} \quad \eta_{nc,2} = \min_{\boldsymbol{\tau}\in\mathcal{W}_D} \|k^{1/2}(\boldsymbol{\tau} - \nabla_h u_{nc})\|_{0,\Omega}. \tag{5.4}$$

Now, a posteriori error estimators for nonconforming elements are defined as follows:

$$\eta^2_{nc,K} = c^2\eta^2_{nc,1,K} + (1-c^2)\eta^2_{nc,2,K} \;\forall K \in \mathcal{T}, \qquad \eta^2_{nc} = c^2\eta^2_{nc,1} + (1-c^2)\eta^2_{nc,2}, \tag{5.5}$$

and

$$\hat\eta^2_{nc,K} = c^2\hat\eta^2_{nc,1,K} + (1-c^2)\hat\eta^2_{nc,2,K} \;\forall K \in \mathcal{T}, \qquad \hat\eta^2_{nc} = c^2\hat\eta^2_{nc,1} + (1-c^2)\hat\eta^2_{nc,2}, \tag{5.6}$$

where $0 < c < 1$ is a parameter to be chosen, e.g., $c^2 = 1/2$. These estimators essentially measure the violations of the continuity of both the normal components of the flux and the tangential derivatives of the true solution on the edges by the nonconforming elements.

Let u_{nc} and $\tilde u_{nc}$ be the solutions of (3.5) with the right-hand sides f and f_h, respectively, where f_h is piecewise constant with $f_h|_K = \frac{1}{|K|}\int_K f\,dx$ for all $K \in \mathcal{T}$. It is easy to see that

$$\|k^{1/2}\nabla_h(u_{nc} - \tilde u_{nc})\|_{0,\Omega} \le C\,\|k^{-1/2}h(f-f_h)\|_{0,\Omega}. \tag{5.7}$$

Let $\boldsymbol\sigma_m \in RT_0$ be the mixed finite element approximation and let $\mathbf{x}_K$ be the center of inertia of K. By the well-known fact [19] that $(\boldsymbol\sigma_m + k\nabla_h\tilde u_{nc})|_K = -\frac12\,f_h|_K(\mathbf{x}-\mathbf{x}_K)$, we have

$$(\boldsymbol\sigma_m + k\nabla_h u_{nc})|_K = -\frac12\,f_h|_K(\mathbf{x}-\mathbf{x}_K) + (k\nabla_h u_{nc} - k\nabla_h\tilde u_{nc})_K.$$

Hence, to avoid the flux recovery, we may replace $\eta_{nc,1,K}$ by $\eta_{nc,f,K} = \frac12\,\|k^{-1/2}f_h(\mathbf{x}-\mathbf{x}_K)\|_{0,K}$ to obtain the following estimators:

$$\tilde\eta^2_{nc,K} = \eta^2_{nc,f,K} + \eta^2_{nc,2,K} \;\forall K \in \mathcal{T}, \qquad \tilde\eta^2_{nc} = c^2\eta^2_{nc,f} + (1-c^2)\eta^2_{nc,2} \tag{5.8}$$

and

$$\bar\eta^2_{nc,K} = \eta^2_{nc,f,K} + \hat\eta^2_{nc,2,K} \;\forall K \in \mathcal{T}, \qquad \bar\eta^2_{nc} = c^2\eta^2_{nc,f} + (1-c^2)\hat\eta^2_{nc,2}, \tag{5.9}$$

where $\eta_{nc,f} = \frac12\,(\sum_{K\in\mathcal{T}}\|k^{-1/2}f_h(\mathbf{x}-\mathbf{x}_K)\|^2_{0,K})^{1/2}$. Now, it follows from (5.4), (5.7), and the triangle inequality that

$$\eta_{nc} \le \hat\eta_{nc}, \quad \eta_{nc} \le \tilde\eta_{nc} + C\,\|k^{-1/2}h(f-f_h)\|_{0,\Omega}, \text{ and } \eta_{nc} \le \bar\eta_{nc} + C\,\|k^{-1/2}h(f-f_h)\|_{0,\Omega}. \tag{5.10}$$

6. Reliability and efficiency bounds. This section analyzes the estimators introduced in the previous section by establishing the reliability and efficiency bounds with constants independent of the size of jumps. To this end, assume that Hypothesis 2.7 in [6] holds. That is, for any two different subdomains $\bar\Omega_i$ and $\bar\Omega_j$ which share at least one point, there is a connected path passing from $\bar\Omega_i$ to $\bar\Omega_j$ through adjacent subdomains such that the diffusion coefficient $k(x)$ is monotone along this path. This assumption is weakened to the quasi-monotonicity in [24].

It is a standard technique (see, e.g., [4, 14]) to analyze estimators for the mixed and nonconforming elements by using the Helmholtz decomposition (see, e.g., [15]).

Lemma 6.1 (Helmholtz decomposition). *For a vector-valued function $\boldsymbol\tau \in L^2(\Omega)^2$, there exist $\alpha \in H^1_D(\Omega)$ and $\beta \in H^1_N(\Omega)$ such that*

$$\boldsymbol\tau = k(x)\nabla\alpha + \nabla^\perp\beta \quad \textit{and} \quad (k^{-1}\boldsymbol\tau, \boldsymbol\tau) = (k\nabla\alpha, \nabla\alpha) + (k^{-1}\nabla^\perp\beta, \nabla^\perp\beta). \tag{6.1}$$

Note that for all $\alpha \in H^1_D(\Omega)$ and all $\beta \in H^1_N(\Omega)$

$$(\nabla\alpha,\, \nabla^{\perp}\beta) = 0. \tag{6.2}$$

It is also common to use Clément-type interpolation operators (see, e.g., [6, 24]) for establishing the reliability bound of a posteriori error estimators. Following [6], one can define the interpolation operator $\mathcal{J} : L^2(\Omega) \to \mathcal{U}_D$ (see [9] for more details) so that

$$\|v - \mathcal{J}v\|_{0,K} + h_K\|\nabla(v - \mathcal{J}v)\|_{0,K} \le C\, h_K k_K^{-1/2}\|k^{1/2}\nabla v\|_{0,\Delta_K}\ \forall\, v \in H^1_D(\Omega), \tag{6.3}$$

where Δ_K is the union of all elements that share at least one vertex with K, and that

$$|(f,\, v - \mathcal{J}v)| \le C\, H_f\, \|k^{1/2}\nabla v\|_{0,\Omega}\ \forall\, v \in H^1_D(\Omega), \tag{6.4}$$

where H_f is higher order for $f \in L^p(\Omega)$ with $p > 2$. Let $\mathcal{U}_N = \{v \in \mathcal{U} :\ v = 0 \text{ on } \Gamma_N\}$. Similarly, one can define a robust interpolation $\mathcal{J}' : L^2(\Omega) \to \mathcal{U}_N$ so that

$$\|v-\mathcal{J}'v\|_{0,K}+h_K\|\nabla^{\perp}(v-\mathcal{J}'v)\|_{0,K} \le C\, h_K k_K^{1/2}\|k^{-1/2}\nabla^{\perp}v\|_{0,\Delta_K}\ \forall\, v \in H^1(\Omega) \tag{6.5}$$

and that

$$|(f,\, v - \mathcal{J}'v)| \le C\, G_f\, \|k^{-1/2}\nabla^{\perp}v\|_{0,\Omega}\ \forall\, v \in H^1(\Omega), \tag{6.6}$$

where G_f is higher order for $f \in L^p(\Omega)$ with $p > 2$.

6.1. Reliability on mixed elements.

THEOREM 6.2. *The estimator η_m defined in (5.1) satisfies the following global reliability bound:*

$$\|k^{-1/2}E_m\|_{0,\Omega} \le C\left(\eta_m + \|k^{-1/2}h(f - f_h)\|_{0,\Omega} + G_{\nabla_h\times(k^{-1}\boldsymbol{\sigma}_m)}\right). \tag{6.7}$$

If $\mathcal{V}_N = RT_0$, then

$$\|k^{-1/2}E_m\|_{0,\Omega} \le C\left(\eta_m + \|k^{-1/2}h(f - f_h)\|_{0,\Omega}\right). \tag{6.8}$$

Proof. Let $E_m = \boldsymbol{\sigma} - \boldsymbol{\sigma}_m = -k\nabla u - \boldsymbol{\sigma}_m \in H_N(\mathrm{div};\Omega)$; by Lemma 6.1 and (6.1), there exist $\alpha_m \in H^1_D(\Omega)$ and $\beta_m \in H^1_N(\Omega)$ such that

$$E_m = k\nabla\alpha_m + \nabla^{\perp}\beta_m \quad \text{and} \quad \|k^{-1/2}E_m\|^2_{0,\Omega} = \|k^{1/2}\nabla\alpha_m\|^2_{0,\Omega} + \|k^{-1/2}\nabla^{\perp}\beta_m\|^2_{0,\Omega}. \tag{6.9}$$

The upper bound of the first term in (6.9) follows easily from (6.2), integration by parts, $E_m \cdot \mathbf{n} = 0$ on Γ_N, $\alpha_m = 0$ on Γ_D, the first equation in (3.3), and the Cauchy–Schwarz inequality that

$$\begin{aligned}\|k^{1/2}\nabla\alpha_m\|^2_{0,\Omega} &= (E_m,\, \nabla\alpha_m) = -(\nabla\cdot E_m,\, \alpha_m) = (\nabla\cdot E_m,\, Q_h\alpha_m - \alpha_m)\\ &= (f - f_h,\, Q_h\alpha_m - \alpha_m) \le C\,\|k^{-1/2}h(f - f_h)\|_{0,\Omega}\|k^{1/2}\nabla\alpha_m\|_{0,\Omega},\end{aligned}$$

which implies

$$\|k^{1/2}\nabla\alpha_m\|^2_{0,\Omega} \le C\,\|k^{-1/2}h(f - f_h)\|^2_{0,\Omega}. \tag{6.10}$$

To bound the second term in (6.9), $\|k^{-1/2}\nabla^\perp\beta_m\|_{0,\Omega}^2$, notice first that $\nabla^\perp\mathcal{U}_N \subset RT_0$. Then the first equation in (3.3) gives

$$(k^{-1}E_m,\, \nabla^\perp v) = (\nabla\cdot\nabla^\perp v,\, e_m) = 0 \quad \forall\, v\in\mathcal{U}_N. \tag{6.11}$$

By (6.2), the fact that $\mathcal{J}'\beta_m \in \mathcal{U}_N$, (6.11), integration by parts, and boundary conditions of $\boldsymbol{\rho}_m - \nabla u$ on Γ_D and $(I-\mathcal{J}')\beta_m$ on Γ_N, we have

$$\begin{aligned}\|k^{-1/2}\nabla^\perp\beta_m\|_{0,\Omega}^2 &= (k^{-1}E_m,\, \nabla^\perp\beta_m) = (k^{-1}E_m,\, \nabla^\perp(\beta_m - \mathcal{J}'\beta_m))\\ &= \left(\boldsymbol{\rho}_m - \nabla u,\, \nabla^\perp(I-\mathcal{J}')\beta_m\right) - \left(\boldsymbol{\rho}_m + k^{-1}\boldsymbol{\sigma}_m,\, \nabla^\perp(I-\mathcal{J}')\beta_m\right)\\ &= -\left(\nabla\times\boldsymbol{\rho}_m,\, (I-\mathcal{J}')\beta_m\right) - \left(\boldsymbol{\rho}_m + k^{-1}\boldsymbol{\sigma}_m,\, \nabla^\perp(I-\mathcal{J}')\beta_m\right),\end{aligned}$$

which, together with the Cauchy–Schwarz inequality and (6.5), implies

$$\begin{aligned}\|k^{-1/2}\nabla^\perp\beta_m\|_{0,\Omega}^2 &\le -\left(\nabla\times\boldsymbol{\rho}_m,\, (I-\mathcal{J}')\beta_m\right) + C\,\eta_m\,\|k^{-1/2}\nabla^\perp\beta_m\|_{0,\Omega}\\ &= -\left(\nabla_h\times(\boldsymbol{\rho}_m + k^{-1}\boldsymbol{\sigma}_m), (I-\mathcal{J}')\beta_m\right) + \left(\nabla_h\times(k^{-1}\boldsymbol{\sigma}_m), (I-\mathcal{J}')\beta_m\right) + C\eta_m\,\|k^{-1/2}\nabla^\perp\beta_m\|_{0,\Omega}.\end{aligned}$$

Using the Cauchy–Schwartz inequality, the inverse inequality, (6.5), and (6.6), we obtain that

$$\begin{aligned}&\left(\nabla_h\times(\boldsymbol{\rho}_m + k^{-1}\boldsymbol{\sigma}_m),\, (I-\mathcal{J}')\beta_m\right)\\ &\le C\,\|k^{1/2}h\nabla_h\times(\boldsymbol{\rho}_m + k^{-1}\boldsymbol{\sigma}_m)\|_{0,\Omega}\,\|k^{-1/2}h^{-1}(I-\mathcal{J}')\beta_m\|_{0,\Omega} \le C\,\eta_m\,\|k^{-1/2}\nabla^\perp\beta_m\|_{0,\Omega}\end{aligned}$$

and that

$$\left(\nabla_h\times(k^{-1}\boldsymbol{\sigma}_m), (I-\mathcal{J}')\beta_m\right) \le C\,G_{\nabla_h\times(k^{-1}\boldsymbol{\sigma}_m)}\|k^{-1/2}\nabla^\perp\beta_m\|_{0,\Omega_z}.$$

Combining the above three inequalities, dividing the quantity $\|k^{-1/2}\nabla^\perp\beta_m\|_{0,\Omega}$, and squaring on both sides give

$$\|k^{-1/2}\nabla^\perp\beta_m\|_{0,\Omega}^2 \le C\left(\eta_m + G_{\nabla_h\times(k^{-1}\boldsymbol{\sigma}_m)}\right)^2,$$

which, together with (6.9) and (6.10), yields (6.7).

Finally, if $\mathcal{V}_N = RT_0$, then $\nabla_h\times(k^{-1}\boldsymbol{\sigma}_m) = 0$. Hence, (6.8) is a direct consequence of (6.7) and the fact that $G_0 = 0$. This completes the proof of the theorem. ∎

COROLLARY 6.3. *The reliability bounds in* Theorem 6.2 *hold for the explicit error estimator* $\hat{\eta}_m$.

Proof. The corollary is a direct consequence of Theorem 6.2 and (5.3). ∎

6.2. Reliability on nonconforming elements.

THEOREM 6.4. *The estimator* η_{nc} *defined in* (5.5) *satisfies the following global reliability bound:*

$$\|k^{1/2}\nabla_h e_{nc}\|_{0,\Omega} \le C\,(\eta_{nc} + H_f)\,. \tag{6.12}$$

Proof. Let $e_{nc} = u - u_{nc}$; by Lemma 6.1 and (6.1), there exist $\alpha_{nc}\in H_D^1(\Omega)$ and $\beta_{nc}\in H_N^1(\Omega)$ such that

$$k\nabla_h e_{nc} = k\nabla\alpha_{nc} + \nabla^\perp\beta_{nc} \quad\text{and}\quad \|k^{1/2}\nabla_h e\|_{0,\Omega}^2 = \|k^{1/2}\nabla\alpha_{nc}\|_{0,\Omega}^2 + \|k^{-1/2}\nabla^\perp\beta_{nc}\|_{0,\Omega}^2. \tag{6.13}$$

By using (3.6), integrations by parts, boundary conditions on $k\nabla u + \boldsymbol{\sigma}_{nc}$ and $(I - \mathcal{J})\alpha_{nc}$, the Cauchy–Schwarz inequality, the fact that $\nabla_h \cdot (k\nabla_h u_{nc}) = 0$, the inverse inequality, (6.3), and (6.4), we have

$$\begin{aligned}
&\|k^{1/2}\nabla\alpha_{nc}\|_{0,\Omega}^2 = (k\nabla_h e_{nc},\, \nabla\alpha_{nc}) = (k\nabla_h e_{nc},\, \nabla(I-\mathcal{J})\alpha_{nc})\\
&= (k\nabla u + \boldsymbol{\sigma}_{nc},\, \nabla(I-\mathcal{J})\alpha_{nc}) - (\boldsymbol{\sigma}_{nc} + k\nabla_h u_{nc},\, \nabla(I-\mathcal{J})\alpha_{nc})\\
&\le (f - \nabla\cdot\boldsymbol{\sigma}_{nc},\, \alpha_{nc} - \mathcal{J}\alpha_{nc}) + C\,\eta_{nc,1}\,\|k^{1/2}\nabla\alpha_{nc}\|_{0,\Omega}\\
&= (f,\, \alpha_{nc} - \mathcal{J}\alpha_{nc}) - (\nabla_h\cdot(\boldsymbol{\sigma}_{nc} + k\nabla_h u_{nc}),\, \alpha_{nc} - \mathcal{J}\alpha_{nc}) + C\,\eta_{nc,1}\,\|k^{1/2}\nabla\alpha_{nc}\|_{0,\Omega}\\
&\le C\,(\eta_{nc,1} + H_f)\,\|k^{1/2}\nabla\alpha_{nc}\|_{0,\Omega}.
\end{aligned}$$

To bound the second term in (6.13), first by integration by parts and the orthogonality of nonconforming elements, i.e., $\int_e [u_{nc}]\,ds = 0$ for any $e\in\mathcal{E}$, we have

$$(\nabla_h u_{nc},\, \nabla^\perp\mathcal{J}'\beta_{nc}) = \sum_{K\in\mathcal{T}}\int_{e\in\partial K} u_{nc}\,(\nabla^\perp\mathcal{J}'\beta_{nc})\cdot\mathbf{n}\,ds = \sum_{e\in\mathcal{E}}(\nabla^\perp\mathcal{J}'\beta_{nc})\cdot\mathbf{n}\int_e [u_{nc}]\,ds = 0.$$

It then follows from integration by parts, boundary conditions on $\boldsymbol{\rho}_{nc}$ and $(I-\mathcal{J}')\beta_{nc}$, the fact that $\nabla_h\times(k^{-1}\nabla_h u_{nc}) = 0$, the Cauchy–Schwarz and inverse inequalities, and (6.5) that

$$\begin{aligned}
&\|k^{-1/2}\nabla^\perp\beta_{nc}\|_{0,\Omega}^2 = (\nabla_h e,\, \nabla^\perp\beta_{nc}) = (-\nabla_h u_{nc},\, \nabla^\perp(I-\mathcal{J}')\beta_{nc})\\
&= (\boldsymbol{\rho}_{nc} - \nabla_h u_{nc},\, \nabla^\perp(I-\mathcal{J}')\beta_{nc}) - (\boldsymbol{\rho}_{nc},\, \nabla^\perp(I-\mathcal{J}')\beta_{nc})\\
&= (\boldsymbol{\rho}_{nc} - \nabla_h u_{nc},\, \nabla^\perp(I-\mathcal{J}')\beta_{nc}) + (\nabla_n\times(\boldsymbol{\rho}_{nc} - \nabla_h u_{nc}),\, (I-\mathcal{J}')\beta_{nc})\\
&\le C\,\eta_{nc,2}\,\|k^{-1/2}\nabla^\perp\beta_{nc}\|_{0,\Omega}.
\end{aligned}$$

Now, the reliability bound in (6.12) is a direct consequence of the above two inequalities. This completes the proof of the theorem. ∎

COROLLARY 6.5. *Let η denote the estimators $\tilde\eta_{nc}$, $\hat\eta_{nc}$, or $\bar\eta_{nc}$, and then we have the following reliability bound:*

$$\|k^{1/2}\nabla_h e_{nc}\|_{0,\Omega} \le C\left(\eta + H_f + \|k^{-1/2}h(f - f_h)\|_{0,\Omega}\right).$$

Proof. The corollary is a direct consequence of Theorem 6.4 and (5.10). ∎

6.3. Efficiency. To establish the efficiency bounds, we make use of the known result on the edge error estimators in [11]. To this end, for any $e\in\mathcal{E}_\Omega$ and any vector-valued function $\boldsymbol{\tau}$ that is piecewise linear with respect to the triangulation $\mathcal{T}$, denote the jump of the normal and tangential components of $\boldsymbol{\tau}$ across $e = K_e^+\cap K_e^-$ by

$$J_{\mathbf{n},e}(\boldsymbol{\tau}) = [\boldsymbol{\tau}\cdot\mathbf{n}_e] = (\boldsymbol{\tau}|_{K_e^+} - \boldsymbol{\tau}|_{K_e^-})\cdot\mathbf{n}_e \quad\text{and}\quad J_{\mathbf{t},e}(\boldsymbol{\tau}) = [\boldsymbol{\tau}\cdot\mathbf{t}_e] = (\boldsymbol{\tau}|_{K_e^+} - \boldsymbol{\tau}|_{K_e^-})\cdot\mathbf{t}_e,$$

respectively. For any $e\in\mathcal{E}\backslash\mathcal{E}_\Omega$, we set

$$J_{\mathbf{n},e}(\boldsymbol{\tau}) = 0 \quad\text{and}\quad J_{\mathbf{t},e}(\boldsymbol{\tau}) = 0.$$

6.3.1. Efficiency on mixed elements. We define an edge error estimator for mixed elements as follows:

$$\eta_{m,E} := \left(\sum_{e\in\mathcal{E}} \eta_{m,e}^2\right)^{1/2} \quad \text{with} \quad \eta_{m,e} = \left(\frac{k_{K_e^+} + k_{K_e^-}}{2}\, h_e \int_e |J_{\mathbf{t},e}(k^{-1}\boldsymbol{\sigma}_m)|^2\, ds\right)^{1/2}. \tag{6.14}$$

PROPOSITION 6.6. *There exists a constant $C > 0$ such that*

$$\eta_{m,e} \le C\, \|k^{-1/2}\nabla^\perp \beta_m\|_{0,\omega_e} \quad \textit{and} \quad \eta_{m,E} \le C\, \|k^{-1/2}\nabla^\perp \beta_m\|_{0,\Omega} \le C\, \|k^{-1/2}E_m\|_{0,\Omega}, \tag{6.15}$$

where ω_e is the union of all elements that share edge e.

Proof. A similar proof as those of Lemmas 6.1 and 6.2 in [10] shows that there exists a constant $C > 0$ independent of k such that

$$h_e \int_e |J_{\mathbf{t},e}(k^{-1}\boldsymbol{\sigma}_m)|^2\, ds \le C\, \|k^{-1}\nabla^\perp \beta_m\|^2_{0,\omega_e}.$$

Hence,

$$\begin{aligned}
\eta_{m,e}^2 &\le C\,(k_{K_e^+} + k_{K_e^-})\|k^{-1}\nabla^\perp\beta_m\|^2_{0,\omega_e}\\
&= C\,(k_{K_e^+} + k_{K_e^-})(k^{-1}_{K_e^+}\|k^{-1/2}\nabla^\perp\beta_m\|^2_{0,K_e^+} + k^{-1}_{K_e^-}\|k^{-1/2}\nabla^\perp\beta_m\|^2_{0,K_e^-})\\
&\le C\,\|k^{-1/2}\nabla^\perp\beta_m\|^2_{0,\omega_e},
\end{aligned}$$

which proves the first inequality in (6.15). Summing it over all edges $e \in \mathcal{E}$ leads to the second inequality in (6.15). ∎

THEOREM 6.7. *The following local efficiency bound for the explicit error estimator $\hat\eta_{m,K}$ holds:*

$$\hat\eta_{m,K}^2 \le C\, \|k^{-1/2}\nabla^\perp\beta_m\|_{0,\omega_K}, \tag{6.16}$$

where ω_K is the union of elements sharing a common edge with K. The following global efficiency bound holds for all error estimators:

$$\eta_m \le \hat\eta_m \le C\, \|k^{-1/2}\nabla^\perp\beta_m\|_{0,\Omega} \le C\, \|k^{1/2}E_m\|_{0,\Omega}. \tag{6.17}$$

Proof. (6.17) follows straightforward from (5.3) and (6.16). To show the validity of (6.16), by Proposition 6.6 it suffices to prove that for any element $K \in \mathcal{T}$

$$\hat\eta_{m,K}^2 \le C \sum_{\epsilon\in\partial K} \eta_{m,e}^2 = C \sum_{e\in\partial K\setminus\partial\Omega} \frac{h_e}{2}\,(k_{K_e^+} + k_{K_e^-}) \int_e |J_{t,e}(k^{-1}\boldsymbol{\sigma}_m)|^2\, ds. \tag{6.18}$$

To do so, for any edge $e \in \partial K$, without loss of generality let K be K_e^+ and let K_e^- be the adjacent element with the common edge e. Since $\boldsymbol{\tau} = k^{-1}\boldsymbol{\sigma}_m$ is piecewise linear, we have

$$\boldsymbol{\tau}|_K = \sum_{e\in\partial K} \left(\tau_{e,1,K}\,\boldsymbol{\psi}_{e,1} + \tau_{e,2,K}\,\boldsymbol{\psi}_{e,2}\right),$$

where $\tau_{e,i,K} = \int_e s^{i-1}(\boldsymbol{\tau}\cdot\mathbf{t}_e)_K\, ds$ is the $(i-1)$th moment of the tangential component of $\boldsymbol{\tau}$ on e and $\boldsymbol{\psi}_{e,i}$ is the nodal basis function of ND_2. For any $\mathbf{x}\in K$,

$$
\begin{aligned}
\hat{\boldsymbol{\rho}}_m + \boldsymbol{\tau} &= \sum_{e\in\partial K} (\hat{\rho}_{e,1} - \tau_{e,1,K})\, \boldsymbol{\psi}_{e,1} + (\hat{\rho}_{e,2} - \tau_{e,2,K})\, \boldsymbol{\psi}_{e,2} \\
&= \sum_{e\in\partial K\setminus\partial\Omega} (1-\gamma_{1,e}) \left(\left(\tau_{e,1,K_e^-} - \tau_{e,1,K}\right) \boldsymbol{\psi}_{e,1} + \left(\tau_{e,2,K_e^-} - \tau_{e,2,K}\right) \boldsymbol{\psi}_{e,2}\right) \\
&= \sum_{e\in\partial K\setminus\partial\Omega} (\gamma_{1,e} - 1) \left(\boldsymbol{\psi}_{e,1} \int_e J_{\mathbf{t},e}(\boldsymbol{\tau}) ds + \boldsymbol{\psi}_{e,2} \int_e s J_{\mathbf{t},e}(\boldsymbol{\tau}) ds \right) \\
&= \sum_{e\in\partial K\setminus\partial\Omega} (\gamma_{1,e} - 1) \left(j_{e,1}\, \boldsymbol{\psi}_{e,1} + j_{e,2}\, \boldsymbol{\psi}_{e,2}\right)
\end{aligned}
$$

with $j_{e,i} = \int_e s^{i-1} J_{\mathbf{t},e}(\boldsymbol{\tau}) ds$. Since $J_{\mathbf{t},e}(\boldsymbol{\tau})$ is a linear function on e, Lemma 4.2 gives

$$
\begin{gathered}
J_{\mathbf{t},e}(\boldsymbol{\tau}) = j_{e,1}\,(\boldsymbol{\psi}_{e,1}\cdot\mathbf{t}_e) + j_{e,2}\,(\boldsymbol{\psi}_{e,2}\cdot\mathbf{t}_e) \ \text{ and} \\
\int_e |J_{\mathbf{t},e}(\boldsymbol{\tau})|^2 ds = j_{e,1}^2 \int_e |\boldsymbol{\psi}_{e,1}|^2 ds + j_{e,2}^2 \int_e |\boldsymbol{\psi}_{e,2}|^2 ds.
\end{gathered}
$$

Now, by the triangle inequality and the fact that $\int_K |\boldsymbol{\psi}_{e,i}|^2 dx \le C h_e \int_e |\boldsymbol{\psi}_{e,i}|^2 ds$, we have

$$
\begin{aligned}
\hat{\eta}_{m,K} &= \|k^{1/2}(\hat{\boldsymbol{\rho}}_m + \boldsymbol{\tau})\|_{0,K} \\
&\le C \left(k_K \sum_{e\in\partial K\setminus\partial\Omega} (1-\gamma_{1,e}) \left(j_{e,1}^2 \int_K |\boldsymbol{\psi}_{e,1}|^2\, dx + j_{e,2}^2 \int_K |\boldsymbol{\psi}_{e,2}|^2\, dx \right)\right)^{1/2} \\
&\le C \left(k_K \sum_{e\in\partial K\setminus\partial\Omega} h_e \int_e |J_{t,e}(\boldsymbol{\tau})|^2\, ds \right)^{1/2} \le C \sum_{\epsilon\in\partial K\setminus\partial\Omega} \eta_{m,e}^2.
\end{aligned}
$$

This proves (6.18) and, hence, the theorem. ◻

6.3.2. Efficiency on nonconforming elements. A weighted edge error estimator for nonconforming elements is defined by

$$
\eta_{nc,E} := \left(\sum_{e\in\mathcal{E}} \eta_{nc,e}^2 \right)^{1/2}
$$

with

$$
\eta_{nc,e}^2 = \frac{2h_e}{k_{K_e^+} + k_{K_e^-}} \int_e |J_{\mathbf{n},e}(k\nabla_h u_{nc})|^2\, ds + \frac{h_e\, k_{K_e^+}\, k_{K_e^-}}{k_{K_e^+} + k_{K_e^-}} \int_e |J_{\mathbf{t},e}(\nabla_h u_{nc})|^2\, ds. \tag{6.19}
$$

THEOREM 6.8. *There exists a constant $C > 0$ such that*

$$
\eta_{nc,e}^2 \le C \left(\|k^{1/2}\nabla_h e_{nc}\|_{0,\omega_e}^2 + \sum_{K\in\mathcal{T}\cap\omega_e} \frac{h_K^2}{k_K} \|f - f_h\|_{0,K}^2 \right) \tag{6.20}
$$

and

$$\eta_{nc,E} \le C\,\|k^{1/2}\nabla_h e_{nc}\|_{0,\Omega} + C\,\|hk^{-1/2}(f-f_h)\|_{0,\Omega}. \tag{6.21}$$

Proof. A similar proof as that for the conforming linear elements in [24] shows that the first term of $\eta^2_{nc,e}$ defined in (6.19) satisfies (6.20). It then suffices to show that (6.20) holds for the second term of $\eta^2_{nc,e}$. To do so, we recall the following inequality (estimate (3.3) in [12]):

$$h_e \int_e |J_{\mathbf{t},e}(\nabla_h u_{nc})|^2\,ds \le C\,\|\nabla_h e_{nc}\|^2_{0,\omega_e}, \tag{6.22}$$

which holds with a constant $C > 0$ independent of the jump of k. Hence,

$$\frac{h_e\, k_{K_e^+}\, k_{K_e^-}}{k_{K_e^+} + k_{K_e^-}} \int_e |J_{\mathbf{t},e}(\nabla_h u_{nc})|^2\,ds \le C \frac{k_{K_e^+}\, k_{K_e^-}}{k_{K_e^+} + k_{K_e^-}} \|\nabla_h e_{nc}\|^2_{0,\omega_e}$$

$$\le C\left(\frac{k_{K_e^-}}{k_{K_e^+} + k_{K_e^-}}\|k^{1/2}\nabla_h e\|^2_{0,K_e^+} + \frac{k_{K_e^+}}{k_{K_e^+} + k_{K_e^-}}\|k^{1/2}\nabla_h e_{nc}\|^2_{0,K_e^-}\right) \le C\,\|k^{1/2}\nabla_h e_{nc}\|^2_{0,\omega_e}.$$

This proves (6.20). The global bound (6.21) is a direct result of (6.20). ∎

THEOREM 6.9. *Let ω_K be the union of elements sharing a common edge with K, and then the explicit error estimator $\hat{\eta}_{nc,K}$ has the following local efficiency bound:*

$$\hat{\eta}^2_{nc,K} \le C\left(\|k^{1/2}\nabla_h e_{nc}\|^2_{0,\omega_K} + \sum_{T\in\omega_K}\frac{h_T^2}{k_T}\|f-f_h\|^2_{0,T}\right). \tag{6.23}$$

For the estimators η_{nc} and $\tilde{\eta}_{nc}$, we have the following global efficiency bound:

$$\eta_{nc},\, \tilde{\eta}_{nc} \le C\,\|k^{1/2}\nabla_h e_{nc}\|_{0,\Omega} + C\,\|hk^{-1/2}(f-f_h)\|_{0,\Omega}. \tag{6.24}$$

Proof. (6.24) is a direct consequence of (5.10) and (6.23). To show the validity of (6.23), by Theorem 6.8 it suffices to prove that for any element $K \in \mathcal{T}$

$$\hat{\eta}^2_{nc,K} \le C \sum_{e\in\partial K} \eta^2_{nc,e}. \tag{6.25}$$

To this end, for any edge $e \in \partial K$, without loss of generality let $\mathbf{n}_e$ be the outward unit vector normal to ∂K and denote by K_e the adjacent element with the common edge e. Let $\boldsymbol{\tau}_2 = -k\nabla_h u_{nc}$. Lemma 4.1 implies $\boldsymbol{\tau}_2|_K = \sum_{e\in\partial K}\tau_{2,e,K}\,\boldsymbol{\phi}_e$. Hence, by (4.12) and (4.13)

$$\hat{\boldsymbol{\sigma}}_{nc} - \boldsymbol{\tau}_2 = \sum_{e\in\partial K}\left(\hat{\sigma}_e - \tau_{2,e,K}\right)\boldsymbol{\phi}_e = \sum_{e\in\partial K\setminus\partial\Omega}\left(\gamma_{2,e}-1\right)\left(\tau_{2,e,K} - \tau_{2,e,K_e}\right)\boldsymbol{\phi}_e$$

$$= \sum_{e\in\partial K\setminus\partial\Omega}(\gamma_{2,e}-1)J_{\mathbf{n},e}(\boldsymbol{\tau}_2)\,\boldsymbol{\phi}_e$$

for any $\mathbf{x} \in K$. Similarly, for $\boldsymbol{\tau}_3 = -\nabla_h u_{nc}$ and any $\mathbf{x} \in K$, we have

$$\hat{\boldsymbol{\rho}}_{nc} - \boldsymbol{\tau}_3 = \sum_{e\in\partial K}\left(\hat{\rho}_e - \tau_{3,e,K}\right)\boldsymbol{\psi}_e(\mathbf{x}) = \sum_{e\in\partial K\setminus\partial\Omega}\left(\gamma_{3,e}-1\right)\left(\tau_{3,e,K} - \tau_{3,e,K_e}\right)\boldsymbol{\psi}_e$$

$$= \sum_{e\in\partial K\setminus\partial\Omega}(\gamma_{3,e}-1)J_{\mathbf{t},e}(\boldsymbol{\tau}_3)\,\boldsymbol{\psi}_e.$$

Now, it follows from the triangle inequality, the facts that

$$\int_K |\boldsymbol{\phi}_e|^2\,dx \le C\,h_e^2 \quad \text{and} \quad \int_K |\boldsymbol{\psi}_e|^2\,dx \le C\,h_e^2,$$

and (4.16) that

$$\begin{aligned}
&\hat{\eta}^2_{nc,K} = c^2\|k^{-1/2}\,(\hat{\boldsymbol{\sigma}}_{nc} - \boldsymbol{\tau}_2)\,\|^2_{0,K} + (1-c^2)\|k^{1/2}\,(\hat{\boldsymbol{\rho}}_{nc} - \boldsymbol{\tau}_3)\,\|^2_{0,K} \\
&\le C\,\left(k_K^{-1}\|\hat{\boldsymbol{\sigma}}_{nc} - \boldsymbol{\tau}_2\|^2_{0,K} + k_K\|\hat{\boldsymbol{\sigma}}_{nc} - \boldsymbol{\tau}_3\|^2_{0,K}\right) \\
&\le C\left(\sum_{e\in\partial K\setminus\partial\Omega} 2k_K^{-1}(1-\gamma_{2,e})^2 h_e^2 |J_{\mathbf{n},e}(\boldsymbol{\tau}_2)|^2 + \sum_{e\in\partial K\setminus\partial\Omega} k_K(1-\gamma_{3,e})^2 h_e^2|J_{\mathbf{t},e}(\boldsymbol{\tau}_3)|^2\right) \\
&= C\left(\sum_{e\in\partial K\setminus\partial\Omega} \frac{h_e}{\left(k_K^{1/2}+k_{K_e}^{1/2}\right)^2}\left(2\int_e |J_{\mathbf{n},e}(\boldsymbol{\tau}_2)|^2\,ds + k_K\,k_{K_e}\int_e |J_{\mathbf{t},e}(\boldsymbol{\tau}_3)|^2\,ds\right)\right) \\
&\le C\left(\sum_{e\in\partial K\setminus\partial\Omega} \frac{h_e}{k_K+k_{K_e}}\left(2\int_e |J_{\mathbf{n},e}(\boldsymbol{\tau}_2)|^2 ds + k_K\,k_{K_e}\int_e |J_{\mathbf{t},e}(\boldsymbol{\tau}_3)|^2\,ds\right)\right) = C\left(\sum_{e\in\partial K\setminus\partial\Omega} \eta^2_{nc,e}\right),
\end{aligned}$$

which proves (6.25) and, hence, (6.23). This completes the proof of the theorem. □

THEOREM 6.10. *There exists a positive constant C such that*

$$\bar{\eta}_{nc} \le C\left(\|k^{1/2}\nabla_h e_{nc}\|_{0,\Omega} + \|k^{1/2}h(f-f_h)\|_{0,\Omega}\right). \tag{6.26}$$

Proof. (6.26) is a direct consequence of Theorem 6.9 and the following inequality (Theorem 3.2 in [1]):

$$\hat{\eta}_{nc,f,K} \le C\left(\|k^{1/2}\nabla\alpha_{nc}\|_{0,K} + \|k^{-1/2}h(f-f_h)\|_{0,K}\right). \qquad \square$$

7. Numerical experiments. In this section, we report some numerical results for an interface problem with intersecting interfaces used by many authors, e.g., [20], which is considered as a benchmark test problem. For this test problem, we show numerically that the recovery-based a posteriori error estimators introduced in [11] for both mixed and nonconforming elements overrefine regions along the interfaces and, hence, fail to reduce the global error. For the same test problem, numerical results show that the estimators introduced in this paper are accurate and generate meshes with optimal decay of the error with respect to the number of unknowns.

To this end, let $\Omega = (-1,1)^2$ and

$$u(r,\theta) = r^\beta \mu(\theta)$$

in the polar coordinates at the origin with $\mu(\theta)$ being a smooth function of θ [20, 9]. The function $u(r,\theta)$ satisfies the interface equation in (2.1) with $\Gamma_N = \emptyset$, $f = 0$, and

$$k(x) = \begin{cases} R & \text{in } (0,1)^2 \cup (-1,0)^2, \\ 1 & \text{in } \Omega \setminus ([0,1]^2 \cup [-1,0]^2). \end{cases}$$

The β depends on the size of the jump. For example, $\beta = 0.1$ is corresponding to $R \approx 161$.

Remark 7.1. This problem does not satisfy Hypothesis 2.7 in [6] and the distribution of its coefficients is not quasi-monotone.

Note that the solution $u(r,\theta)$ is only in $H^{1+\beta-\epsilon}(\Omega)$ for any $\epsilon > 0$ and, hence, it is very singular for small β at the origin. This suggests that refinement is centered around the origin. In this example we choose $\tilde{\eta}_{nc}$ and $\bar{\eta}_{nc}$ with constant $c^2 = 0.5$ as the implicit and explicit error estimators for the nonconforming method, respectively.

Starting with a coarse triangulation $\mathcal{T}_0$, a sequence of meshes is generated by using a standard adaptive meshing algorithm that adopts the Dörfler's bulk marking strategy: construct a *minimal* subset $\hat{\mathcal{T}}$ of $\mathcal{T}$ such that

$$\sum_{K\in\hat{\mathcal{T}}} \eta_K^2 \geq \theta_E^2 \sum_{K\in\mathcal{T}} \eta_K^2 \tag{7.1}$$

with $\theta_E = 0.2$ which is not critical for better performance. Marked triangles are refined regularly by dividing each into four congruent triangles. Additionally, irregularly refined triangles are needed in order to make the triangulation admissible.

Define the effectivity index:

$$\text{eff-index}_{\text{m}} := \frac{\eta}{\|k^{1/2}\nabla u + k^{-1/2}\boldsymbol{\sigma}_m\|_{0,\Omega}} \quad \text{and} \quad \text{eff-index}_{\text{nc}} := \frac{\eta}{\|k^{1/2}\nabla_h(u-u_{nc})\|_{0,\Omega}},$$

and use the following stopping criteria:

$$\text{rel-err}_{\text{m}} := \frac{\|k^{1/2}\nabla u + k^{-1/2}\boldsymbol{\sigma}_m\|_{0,\Omega}}{\|k^{-1/2}\boldsymbol{\sigma}\|_{0,\Omega}} \leq \text{tol} \quad \text{and} \quad \text{rel-err}_{\text{nc}} := \frac{\|k^{1/2}\nabla_h(u-u_{nc})\|_{0,\Omega}}{\|k^{1/2}\nabla u\|_{0,\Omega}} \leq \text{tol}.$$

Denote by l the number of levels of refinement and by N the number of vertices of triangulation.

The error estimators introduced in [11] for both mixed and nonconforming finite element approximations to the Poisson equations recover the gradient in the continuous linear finite element space. A natural extension of these estimators to the interface problems is to recover either the gradient or the flux again in the continuous linear finite element space. More specifically, for the mixed method, let $\boldsymbol{\sigma}_m$ be the solution of (3.1) and let $\boldsymbol{\rho}_{m,f} \in \mathcal{U}^2$ and $\boldsymbol{\rho}_{m,g} \in \mathcal{U}^2$ satisfy the following problems:

$$(k^{-1}\boldsymbol{\rho}_{m,f},\, \boldsymbol{\tau}) = (k^{-1}\boldsymbol{\sigma}_m,\, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathcal{U}^2$$
$$\text{and} \quad (k\boldsymbol{\rho}_{m,g},\, \boldsymbol{\tau}) = -(\boldsymbol{\sigma}_m,\, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathcal{U}^2,$$

respectively. Then the corresponding error estimators are defined by

$$\eta_{m,CB,f} = \|k^{-1/2}(\boldsymbol{\sigma}_m - \boldsymbol{\rho}_{m,f})\|_{0,\Omega} \quad \text{and} \quad \eta_{m,CB,g} = \|k^{-1/2}\boldsymbol{\sigma}_m + k^{1/2}\boldsymbol{\rho}_{m,g}\|_{0,\Omega}.$$

For the nonconforming method, let u_{nc} be the solution of (3.5) and let $\boldsymbol{\rho}_{nc,f} \in \mathcal{U}^2$ and $\boldsymbol{\rho}_{nc,g} \in \mathcal{U}^2$ satisfy the following problems:

$$(k^{-1}\boldsymbol{\rho}_{nc,f},\, \boldsymbol{\tau}) = (-\nabla_h u_{nc},\, \boldsymbol{\tau}) \quad \text{and} \quad (k\boldsymbol{\rho}_{nc,g},\, \boldsymbol{\tau}) = (k\nabla_h u_{nc},\, \boldsymbol{\tau})$$

for all $\boldsymbol{\tau} \in \mathcal{U}^2$, respectively. Then the corresponding error estimators are defined by

$$\eta_{nc,CB,f} = \|k^{1/2}\nabla_h u_{nc} + k^{-1/2}\boldsymbol{\rho}_{nc,f}\|_{0,\Omega} \quad \text{and} \quad \eta_{nc,CB,g} = \|k^{1/2}(\nabla_h u_{nc} - \boldsymbol{\rho}_{nc,g})\|_{0,\Omega}.$$

TABLE 1
Comparison of estimators for relative error less than 0.1 *for mixed methods.*

	l	N	err	rel-err	η	eff-index
η_m	66	3329	0.0558	0.0988	0.0623	1.1170
$\hat{\eta}_m$	69	2693	0.0560	0.0992	0.0815	1.4537
$\eta_{m,CB,f}$	96	7169	0.0557	0.0986	0.0731	1.3110
$\eta_{m,CB,g}$	83	4021	0.0556	0.0984	0.1845	3.3208

TABLE 2
Comparison of estimators for relative error less than 0.1 *for nonconforming methods.*

	l	N	err	rel-err	η	eff-index
η_{nc}	70	3343	0.0479	0.0982	0.0479	0.8636
$\bar{\eta}_{nc}$	72	2780	0.0557	0.0985	0.0566	1.0170
$\hat{\eta}_{nc}$	71	2960	0.0563	0.0997	0.0683	1.2133
$\eta_{nc,CB,f}$	97	7090	0.0562	0.0995	0.0736	1.3083
$\eta_{nc,CB,g}$	102	6369	0.0562	0.0995	0.0780	1.3879

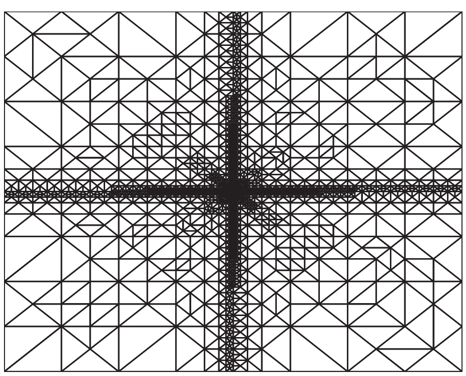

FIG. 1. *Mesh generated by* $\eta_{m,CB,f}$.

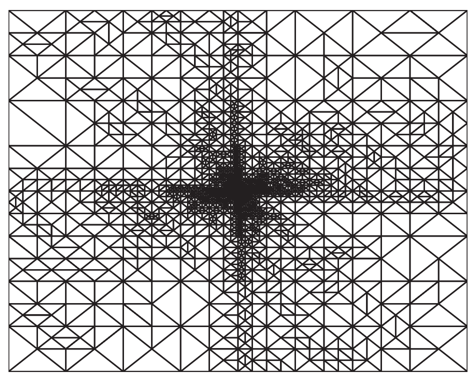

FIG. 2. *Mesh generated by* $\eta_{m,CB,g}$.

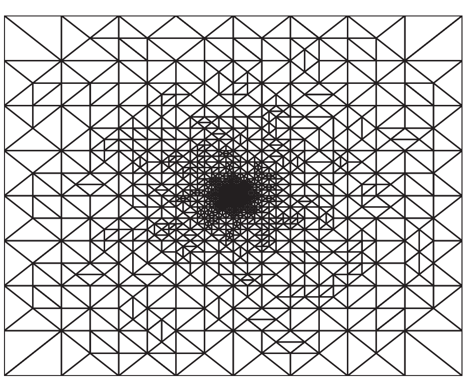

FIG. 3. *Mesh generated by* η_m.

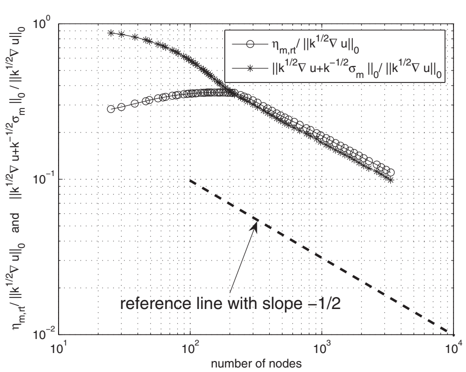

FIG. 4. *Error and estimator* η_m.

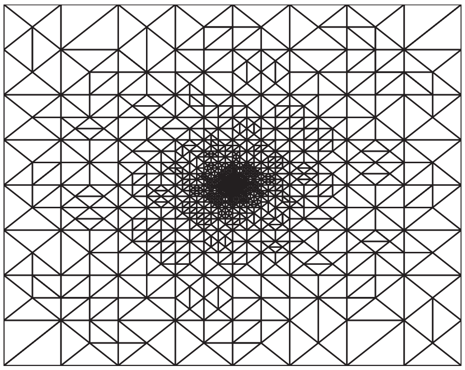

FIG. 5. *Mesh generated by* $\hat{\eta}_m$.

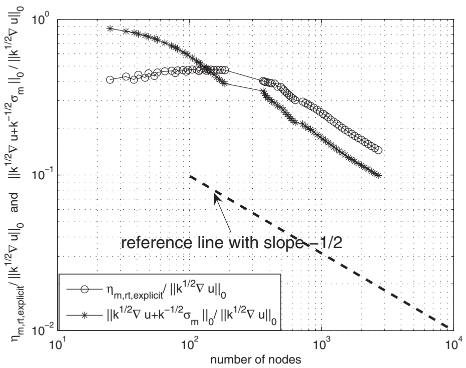

FIG. 6. *Error and estimator* $\hat{\eta}_m$.

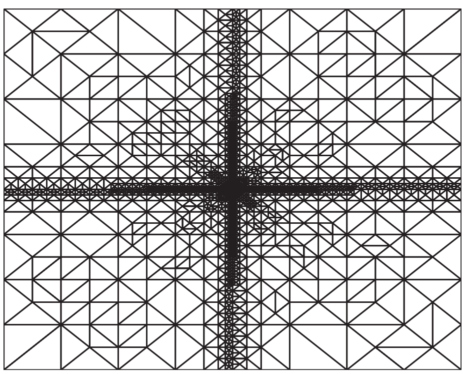

FIG. 7. *Mesh generated by* $\eta_{nc,CB,f}$.

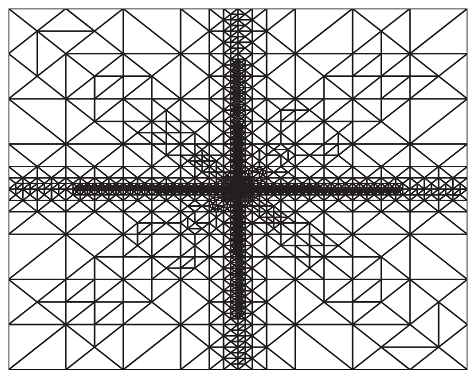

FIG. 8. *Mesh generated by* $\eta_{nc,CB,g}$.

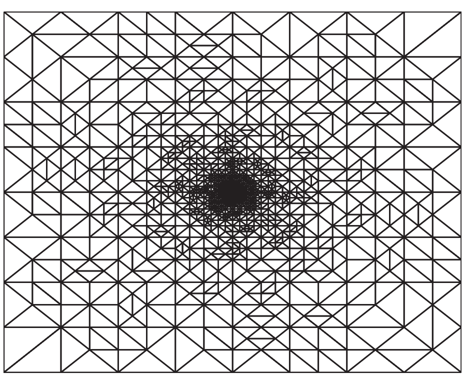

FIG. 9. *Mesh generated by* η_{nc}.

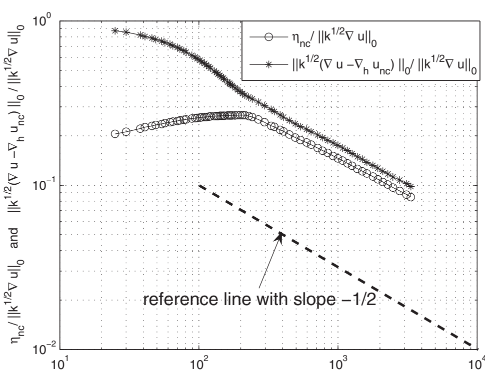

FIG. 10. *Error and estimator* η_{nc}.

We start with the coarsest triangulation $\mathcal{T}_0$ obtained from halving 16 congruent squares by connecting the bottom left and upper right corners. We report numerical results with the stopping criterion tol $= 0.1$. From Tables 1 and 2, and Figures 1, 2, 7, and 8, it is clear that the CB estimators introduce unnecessary refinements along the interfaces. Meshes generated by η_m, $\hat{\eta}_m$, η_{nc}, $\hat{\eta}_{nc}$, and $\bar{\eta}_{nc}$ (shown in Figures 3, 5, 9, 11, and 13) are similar. The comparisons of errors and estimators are depicted

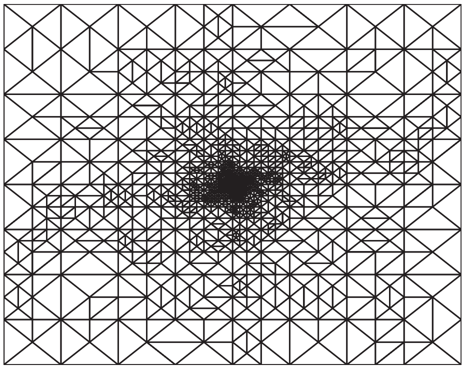

FIG. 11. *Mesh generated by $\hat{\eta}_{nc}$.*

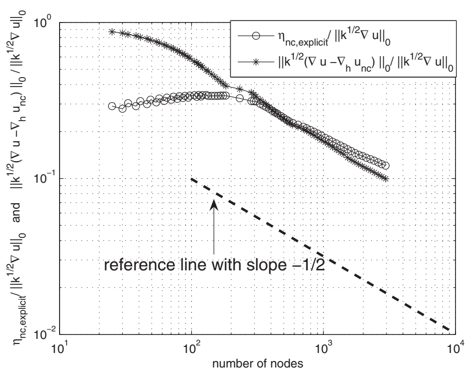

FIG. 12. *Error and estimator $\hat{\eta}_{nc}$.*

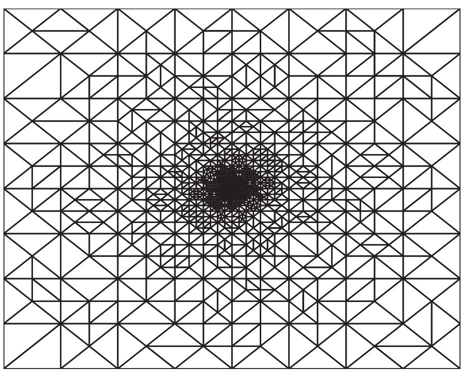

FIG. 13. *Mesh generated by $\bar{\eta}_{nc}$.*

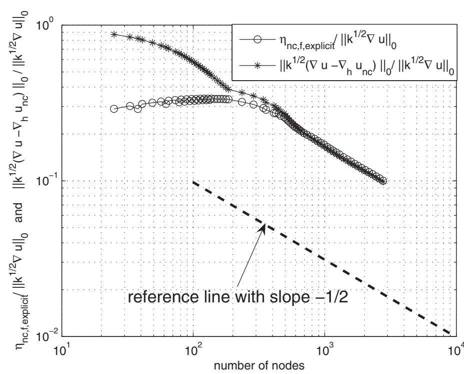

FIG. 14. *Error and estimator $\bar{\eta}_{nc}$.*

in Figures 4, 6, 10, 12, and 14. By inspecting the effectivity index, all the error estimators introduced in this paper are accurate. Moreover, the slope of the log(dof)-log(relative error) for all estimators is $-1/2$, which indicates the optimal decay of the error with respect to the number of unknowns.

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