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GOAL-ORIENTED LOCAL A POSTERIORI ERROR ESTIMATORS FOR H(div) LEAST-SQUARES FINITE ELEMENT METHODS*

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Abstract. We propose a goal-oriented, local a posteriori error estimator for H(div) least-squares (LS) finite element methods. Our main interest is to develop an a posteriori error estimator for the flux approximation in a preassigned region of interest $D \subset \Omega$. The estimator is obtained from the LS functional by scaling residuals with proper weight coefficients. The weight coefficients are given in terms of local mesh size h_T and a function ω_D depending on the distance to D. This new error estimator measures the pollution effect from the outside region of D and provides a basis for local refinement in order to efficiently approximate the solution in D. Numerical experiments show superior performances of our goal-oriented a posteriori estimators over the standard LS functional and global error estimators.

Key words. finite element methods, a posteriori error estimates, least-squares method

AMS subject classifications. 65M60, 65M15

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1. Introduction.

Self-adaptive discretization methods have gained enormous importance for numerical solutions of partial differential equations [1, 3, 4, 7, 8, 9, 11, 26, 41, 42, 43, 45]. A typical algorithm uses information gained during the current step of the computation to produce a new locally refined mesh for the next iteration and repeats the procedure until the error is less than a target tolerance. The question is how to identify regions for refining the current discretization and how to determine whether the current error of interest is below the target tolerance. The key ingredient is a posteriori error estimates which are able to accurately locate sources of global and local error in the current approximation.

Standard a posteriori error estimators are typically global. However, in many practical problems, phenomenon of interest is much smaller than the typical problem size, and hence it is critical to concentrate the computational effort to guarantee local accuracy in a preassigned region of interest [5, 6, 27, 33]. It is well known that the error in a local region is affected by the error outside of the region, namely the so-called pollution effect. The least-squares (LS) methods are not free from the pollution effect [11, 30]. In order to minimize the computational efforts, it is better to refine the mesh outside the region of interest just enough to prevent negative influence on the error of interest. For the standard Galerkin methods, Liao and Nochetto [33] provided local a posteriori error estimators which account for and quantify the pollution effect in the energy norm by deriving a computable version of local a priori error estimates of Nitsche and Schatz [35].

LS methods have recently gained substantial interest as an alternative to mixed Galerkin methods and have been successfully applied to a wide range of problems

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[2, 12, 13, 14, 15, 16, 18, 19, 20, 21, 34, 36, 37, 44]. The LS method is not subject to the inf-sup condition and the resulting algebraic equation is symmetric and positive definite. An additional advantage of the LS method is that the LS functional can be used effectively as an a posteriori error estimator in the natural energy norm. More specifically, the value of the LS functional at the current approximation is exactly the square of the true error in the natural norm associated with the corresponding LS weak formulation.

There has been tremendous interest in a posteriori error estimators for numerical methods, and several types of a posteriori error estimators have been proposed for the standard and mixed Galerkin methods [3, 4, 7, 8, 9, 33, 41, 42, 43, 45]. On the other hand, research concerning a posteriori error estimators for LS methods are somewhat limited to using the LS functional as an estimator [11, 22, 40]. While the LS functional is an effective estimator for the error in the natural energy norm, that would not be the case in the L_2 norm or on a preassigned region of interest.

The flux variable is a primary quantity of interest in many applications. Recently, new a posteriori error estimators for LS methods have been developed for the flux in the global L_2 norm [31, 32]. The estimators are obtained by using the local mesh size as a weight coefficient scaling the respective residuals in the LS functional. These estimators show a superior performance over the LS functional as global error estimators for the flux in the L_2 norm.

In this paper, we develop goal-oriented a posteriori error estimators for the flux variable $\boldsymbol{\sigma} = -\nabla u$ in a preassigned local area of interest $D \subset \Omega$. The new error estimator is obtained from the LS functional by scaling residuals through proper weight coefficients in terms of the local mesh size h_T and a distance function to D. For the second order elliptic equation $-\triangle u + X\, u = f$, the resulting local a posteriori estimator has the form

$$\eta_T = \left\| \sqrt{\omega^2 + h_T^2}(\boldsymbol{\sigma}_h + \nabla u_h) \right\|_{0,T}^2 + \| h_T(f - \nabla\cdot\boldsymbol{\sigma}_h - X u_h)\|_{0,T}^2$$

$$= \int_T (\omega^2 + h_T^2)(\boldsymbol{\sigma}_h + \nabla u_h)^2 dx + \int_T h_T^2 (f - \nabla\cdot\boldsymbol{\sigma}_h - X u_h)^2 dx,$$

where h_T is the local mesh size of an element T and $\omega_D \in W_\infty^1(\Omega)$ is a weight (distance) function such that $\omega_D(x) = 1$ for $x \in D$ and decreases as x moves away from D. This new estimator provides a basis for a mesh refinement strategy for an efficient algorithm to approximate the flux variable in the local area of interest. Our numerical experiments show superior performances of the estimator over the LS functional. When $D = \Omega$ and the mesh size is small, the estimator becomes almost identical to the global estimator developed in [32, 31]. Due to the local nature of the estimator, we need to use a different technique involving a weight function in order to develop goal-oriented error estimators in a local area of interest. A novel feature of the estimator is the use of the weight function ω_D in the a posteriori error estimator. The pollution effect is measured through the weight function. The choice of ω_D in this paper is heuristic. Rigorous treatments for the choice of weight functions will be left for future study.

Adjoint methods are developed in the a posteriori error analysis where the quantity of interest is a functional of the solution, e.g., local average of the solution or point value of the solution; see [10, 25, 28] and the references therein. The methods involve solving adjoint problems on a finer mesh or on higher order finite element spaces. As a result, the methods tend to be expensive.

This paper is organized as follows: In section 2, we introduce notation and the least-squares method. In section 3, we describe finite element spaces and present some preliminary results. We analyze goal-oriented a posteriori error estimator in section 4 and present numerical experiments in section 5.

2. Some preliminaries. Let Ω be a bounded domain in $\mathbb{R}^n$ ($n = 2$ or 3) with Lipschitz boundary $\bar\Gamma = \bar\Gamma_D \cup \bar\Gamma_N$. For convenience, assume that $\Gamma_D \neq \emptyset$. Consider the boundary value problem

$$-\triangle u + \mathbf{b}\cdot\nabla u + cu = -\triangle u + X\, u = f \qquad \text{in } \Omega, \tag{2.1}$$

with boundary conditions

$$u = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n}\cdot\nabla\, u = 0 \quad \text{on } \Gamma_N. \tag{2.2}$$

Remark 2.1. The extension of results in this paper to problems with diffusion tensor $\mathcal{A}$ is straightforward, where the matrix $\mathcal{A}$ is symmetric, bounded, and uniformly positive definite, i.e., there exist positive constants α_0 and α_1 such that

$$\alpha_0\zeta^T\zeta \le \zeta^T A\,\zeta \le \alpha_1\zeta^T\zeta$$

for all $\zeta \in \mathbb{R}^n$ and all $x \in \overline{\Omega}$.

Assume that (2.1) has a unique solution u. Moreover, assume that u satisfies the following a priori estimate [29]: There exists a positive constant C independent of f such that

$$\|u\|_2 \le C\|f\|_0. \tag{2.3}$$

Hereafter, we use C with or without subscripts in this paper to denote a generic positive constant, possibly different at different occurrences, that is independent of the mesh size h.

By introducing the flux variable $\boldsymbol{\sigma} = -\nabla u$, the original problem may be rewritten as a system of first-order differential equations

$$\begin{aligned} \boldsymbol{\sigma} + \nabla u &= 0 \qquad \text{in } \Omega,\\ \nabla\cdot\boldsymbol{\sigma} + Xu &= f \qquad \text{in } \Omega \end{aligned} \tag{2.4}$$

with boundary conditions

$$u = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n}\cdot\boldsymbol{\sigma} = 0 \quad \text{on } \Gamma_N. \tag{2.5}$$

Let $H^s(\Omega)$ denote the Sobolev space of order s defined on Ω. The norm in $H^s(\Omega)$ will be denoted by $\|\cdot\|_s$. For $s = 0, H^s(\Omega)$ coincides with $L^2(\Omega)$. We shall use spaces

$$V = \{u \in H^1(\Omega) :\ u = 0 \text{ on } \Gamma_D\}$$

$$\text{and} \quad \mathbf{W} \equiv \{\boldsymbol{\sigma} \in (L^2(\Omega))^n : \nabla\cdot\boldsymbol{\sigma} \in L^2(\Omega) \text{ and } \mathbf{n}\cdot\boldsymbol{\sigma} = 0 \text{ on } \Gamma_N\} \subset H(\mathrm{div})$$

with norms

$$\|u\|_1^2 = (u,u) + (\nabla u, \nabla u) \quad \text{and} \quad \|\boldsymbol{\sigma}\|_{H(\mathrm{div})}^2 = (\nabla\cdot\boldsymbol{\sigma}, \nabla\cdot\boldsymbol{\sigma}) + (\boldsymbol{\sigma},\boldsymbol{\sigma}).$$

For $D \subset \Omega$, let

$$\|v\|_{k,D}^2 = \int_D \sum_{\alpha\le k} |D^\alpha v(x)|^2 dx.$$

The following LS functional $J(v, \mathbf{q})$ for the first-order system in (2.4) is proposed in [19, 36]:

$$J(v, \mathbf{q}) = (\nabla \cdot \mathbf{q} + Xv - f\,,\, \nabla \cdot \mathbf{q} + Xv - f) + (\mathbf{q} + \nabla u, \mathbf{q} + \nabla v), \tag{2.6}$$

where $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^n$. The LS minimization problem is as follows: Find $u \in V$ and $\boldsymbol{\sigma} \in \mathbf{W}$ such that

$$J(u, \boldsymbol{\sigma}) = \inf_{v \in V, \mathbf{q} \in \mathbf{W}} J(v, \mathbf{q}).$$

By taking variations in (2.6) with respect to v and $\mathbf{q}$, the weak formulation becomes the following: Find $u \in V$ and $\boldsymbol{\sigma} \in \mathbf{W}$ such that

$$b(u, \boldsymbol{\sigma}; v, \mathbf{q}) = (f, \nabla \cdot \mathbf{q} + Xv) \quad \text{for all } v \in V, \quad \mathbf{q} \in \mathbf{W}, \tag{2.7}$$

where

$$b(u, \boldsymbol{\sigma}; v, \mathbf{q}) = (\nabla \cdot \boldsymbol{\sigma} + Xu, \nabla \cdot \mathbf{q} + Xv) + (\boldsymbol{\sigma} + \nabla u, \mathbf{q} + \nabla v).$$

3. Finite element approximation. Let $\mathcal{T}_h$ be a regular triangulation of Ω (see [23]) with triangular/tetrahedra element of size $h = \max\{diam(K); K \in \mathcal{T}_h\}$. Let $P_k(K)$ be the space of polynomials of degree k on K and denote the local Raviart–Thomas space of order k on K:

$$RT_k(K) = P_k(K)^n + \mathbf{x}P_k(K),$$

with $\mathbf{x} = (x_1, \ldots, x_n)$. Then the standard (conforming) continuous piecewise polynomials of degree k and the standard $H(\text{div})$ conforming Raviart–Thomas space of index r [38] are defined by

$$V_h = \{v \in V : v|_K \in P_k(K) \text{ for all } K \in \mathcal{T}_h\}$$
$$\text{and} \quad \mathbf{W}_h = \{\boldsymbol{\tau} \in \mathbf{W} : \boldsymbol{\tau}|_K \in RT_r(K) \text{ for all } K \in \mathcal{T}_h\},$$

respectively. It is well known (see [23]) that V_h has the following approximation property: Let $k \geq 1$ be an integer and let $l \in [1, k+1]$,

$$\inf_{v_h \in V_h} \|v - v_h\|_1 \leq Ch^l \|v\|_{l+1} \tag{3.1}$$

for $u \in H^{l+1}(\Omega)$. It is also well known (see [38]) that $\mathbf{W}_h$ has the following approximation property: Let $r \geq 0$ be an integer and let $l \in [1, r+1]$,

$$\inf_{\boldsymbol{\tau}_h \in \mathbf{W}_h} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{H(div)} \leq Ch^l (\|\boldsymbol{\tau}\|_l + \|\nabla \cdot \boldsymbol{\tau}\|_l) \tag{3.2}$$

for $\boldsymbol{\tau} \in H^l(\Omega)$ with $\nabla \cdot \boldsymbol{\tau} \in H^l(\Omega)$.

Remark 3.1. Extension of analysis in the paper may be generalized to other conforming finite element spaces of $\mathbf{W}$. However, the Raviart–Thomas elements require less regularity of the approximated function than other elements.

The finite element approximation to (2.7) is as follows: Find $u_h \in V_h$ and $\boldsymbol{\sigma}_h \in \mathbf{W}_h$ such that

$$b(u_h, \boldsymbol{\sigma}_h; v_h, \mathbf{q}_h) = (f, \nabla \cdot \mathbf{q}_h + X\, v_h) \tag{3.3}$$

for all $v_h \in V_h$ and $\mathbf{q}_h \in \mathbf{W}_h$. It is well known that (3.3) has a unique solution since $V_h \subset V$ and $\mathbf{W}_h \subset \mathbf{W}$. Moreover, the error has the orthogonal property

$$b(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h; v_h, \mathbf{q}_h) = 0 \quad \text{for all } v_h \in V_h,\ \mathbf{q}_h \in \mathbf{W}_h. \tag{3.4}$$

To develop a posteriori error estimators for the flux in the L_2 norm, we will use the following error estimate for $\|u - u_h\|_0$ (see [31]).

THEOREM 3.1. *Let* $(u, \boldsymbol{\sigma})$ *satisfy* (2.7) *and let* $(u_h, \boldsymbol{\sigma}_h)$ *satisfy* (3.3). *Then,*

$$\|u - u_h\|_0^2 \le C \left(\sum_{T \in \mathcal{T}_h} \|h_T(\boldsymbol{\sigma}_h + \nabla u_h)\|_{0,T}^2 + \|h_T(f - \nabla \cdot \boldsymbol{\sigma}_h - X\, u_h)\|_{0,T}^2 \right).$$

4. Goal-oriented a posteriori error estimators. In this section, we develop goal-oriented, local a posteriori error estimators for the LS method. Particularly, for a given preassigned region $D \subset \Omega$, we propose an a posteriori error estimator for $\|\sigma - \sigma_h\|_{0,D}$ and $\|\nabla(u - u_h)\|_{0,D}$.

To this end, denote by $\omega_D(x) \in W^1_\infty(\Omega)$ a weight (distance) function such that $\omega_D(x) = 1$ in D and decreases as x moves away from D. This weight function plays an important role for measuring and controlling pollution effect. For the choice of ω_D, see Remark 4.2. To simplify notation, denote the weight function by ω without the subscript D. Let

$$e_h = u - u_h \qquad \text{and} \qquad \mathbf{E}_h = \boldsymbol{\sigma} - \boldsymbol{\sigma}_h;$$

then (2.4) implies

$$\mathbf{E}_h + \nabla e_h = -(\boldsymbol{\sigma}_h + \nabla u_h) \quad \text{and} \quad \nabla \cdot \mathbf{E}_h + X\, e_h = f - \nabla \cdot \boldsymbol{\sigma}_h - X u_h. \tag{4.1}$$

The goal-oriented a posteriori error estimator for the flux is defined by

$$\eta^2 = \sum_{T \in \mathcal{T}_h} \eta_T^2$$

with

$$\eta_T = \left(\left\| \sqrt{\omega^2 + h_T^2}(\boldsymbol{\sigma}_h + \nabla u_h) \right\|_{0,T}^2 + \|h_T(f - \nabla \cdot \boldsymbol{\sigma}_h - X u_h)\|_{0,T}^2 \right)^{1/2} \tag{4.2}$$

for all $T \in \mathcal{T}_h$. Using (4.1), we have

$$\eta_T = \left(\left\| \sqrt{\omega^2 + h_T^2}(\mathbf{E}_h + \nabla e_h) \right\|_{0,T}^2 + \|h_T(\nabla \cdot \mathbf{E}_h + X e_h)\|_{0,T}^2 \right)^{1/2}. \tag{4.3}$$

This estimator provides a basis for identifying the local region for mesh refinements. An adaptive mesh refinement algorithm using this estimator will produce an efficient approximation for the solution in the preassigned area of interest.

Remark 4.1. Note that the first term on the right-hand side of the above equality is about the same strength as the flux error in the L_2 norm when $\omega = 1$, i.e., $T \subset D$. On the other hand, the second term without h_T is the strongest term due to the presence of $\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)$, but it is weakened by the scaling factor h_T.

Remark 4.2. A *unique feature* of the estimator defined in (4.2) is the use of a weight function in order to measure the pollution effect. When a cutoff function is used to isolate a local region of interest, the pollution error from outside the region is reflected in the unknown constant. Here, we use a weight function to reflect the error in the a posteriori error estimator. A common assumption is that the effect of pollution weakens as a factor as the distance of the pollution source increases. However, there is presently no available theory for measuring this effect. In this paper, we choose the following weight function ω_D:

$$\omega_D(x) = \frac{C_D}{\mathrm{dist}(x, D) + C_D}, \tag{4.4}$$

where $C_D = \frac{mea(D)}{mea(\Omega)}$ and $\mathrm{dist}(x, D) = \inf_{y\in D} \|x - y\|$. It is easy to check that this choice of ω_D satisfies the assumptions in Theorem 4.1. The above choice of ω_D is heuristic, and a rigorous treatment is the topic of future study.

The following lemma concerns an upper bound for the residual of the equilibrium equation in the H^{-1} norm.

LEMMA 4.1. *Let* $(u, \boldsymbol{\sigma})$ *satisfy* (2.7) *and let* $(u_h, \boldsymbol{\sigma}_h)$ *satisfy* (3.3). *Then, for all* $v \in V$,

$$\begin{aligned}(\nabla\cdot \mathbf{E}_h + Xe_h\,, v) &\le C\left(\sum_{T\in\mathcal{T}_h} \|h_T(\mathbf{E}_h + \nabla e_h)\|_{0,T}^2 + \|h_T(\nabla\cdot\mathbf{E}_h + Xe_h)\|_{0,T}^2\right)^{1/2} \|v\|_1 \\ &\le C\,\eta\,\|v\|_1. \qquad (4.5)\end{aligned}$$

Proof. For any given $v \in V$, consider an auxiliary problem

$$\begin{aligned}\boldsymbol{\psi} + \nabla z &= 0 \qquad \text{in } \Omega,\\ \nabla\cdot\boldsymbol{\psi} + Xz &= v \qquad \text{in } \Omega\end{aligned} \tag{4.6}$$

with boundary conditions

$$z = 0 \;\text{ on } \Gamma_D \quad \text{ and } \quad \mathbf{n}\cdot\boldsymbol{\psi} = 0 \;\text{ on } \Gamma_N.$$

Using both equations in (4.6), the definition of the bilinear form $b(\cdot,\cdot\,;\cdot,\cdot)$, and orthogonality property (3.4), we have

$$\Big(\nabla\cdot\mathbf{E}_h + X\,e_h\,, v\Big) = b(e_h, \mathbf{E}_h\,;\, z, \boldsymbol{\psi}) = b(e_h, \mathbf{E}_h\,;\, z - z_I, \boldsymbol{\psi} - \boldsymbol{\psi}_I). \tag{4.7}$$

Here, $z_I \in V_h$ and $\boldsymbol{\psi}_I \in \mathbf{W}_h$ are the standard interpolations of z and $\boldsymbol{\psi}$, respectively, which satisfy the following approximation properties [17, 39]:

$$\begin{aligned}&\|z - z_I\|_{1,T} \le C\,h_T\|z\|_{2,N(T)}, \quad \|\boldsymbol{\psi} - \boldsymbol{\psi}_I\|_{0,T} \le Ch_T\|\boldsymbol{\psi}\|_{1,N(T)},\\ &\text{and } \|\nabla\cdot(\boldsymbol{\psi} - \boldsymbol{\psi}_I)\|_{0,T} \le Ch_T\|\nabla\cdot\boldsymbol{\psi}\|_{1,N(T)},\end{aligned}$$

where $N(T) = \cup_{T'\cap T\neq\emptyset} T'$ with $T' \in \mathcal{T}_h$.

It follows from the Cauchy–Schwarz and the triangle inequalities that

$$
\begin{aligned}
&\Big(\mathbf{E}_h + \nabla e_h,\, \boldsymbol{\psi} - \boldsymbol{\psi}_I + \nabla(z - z_I)\Big) \\
&\le C \sum_{T\in\mathcal{T}_h} \|\mathbf{E}_h + \nabla e_h\|_{0,T}\big(\|\boldsymbol{\psi} - \boldsymbol{\psi}_I\|_{0,T} + \|\nabla(z - z_I)\|_{0,T}\big) \\
&\le C \sum_{T\in\mathcal{T}_h} h_T\|\mathbf{E}_h + \nabla e_h\|_{0,T}\big(\|\boldsymbol{\psi}\|_{1,N(T)} + \|z\|_{2,N(T)}\big) \\
&\le C \left(\sum_{T\in\mathcal{T}_h} \|h_T(\mathbf{E}_h + \nabla e_h)\|_{0,T}^2\right)^{1/2} (\|\boldsymbol{\psi}\|_1 + \|z\|_2)\,. \qquad (4.8)
\end{aligned}
$$

Since z satisfies (2.1) and (2.2) with $f = v$, the regularity estimate in (2.3) implies

$$\|z\|_2 \le C\|v\|_0.$$

Combining with both equations in (4.6) and using the triangle inequality, we have

$$\|\boldsymbol{\psi}\|_1 = \|\nabla z\|_1 \le C\|v\|_0 \quad \text{and} \quad \|\nabla\cdot\boldsymbol{\psi}\|_1 \le \|v\|_1 + \|Xz\|_1 \le C\|v\|_1.$$

These inequalities with (4.8) lead to

$$\Big(\mathbf{E}_h + \nabla e_h,\, \boldsymbol{\psi} - \boldsymbol{\psi}_I + \nabla(z - z_I)\Big) \le C \left(\sum_{T\in\mathcal{T}_h} \|h_T(\mathbf{E}_h + \nabla e_h)\|_{0,T}^2\right)^{1/2} \|v\|_1. \qquad (4.9)$$

A similar argument as above yields

$$
\begin{aligned}
&\Big(\nabla\cdot\mathbf{E}_h + X\, e_h\,, \nabla\cdot(\boldsymbol{\psi} - \boldsymbol{\psi}_I) + X(z - z_I)\Big) \qquad (4.10)\\
&\qquad \le C \left(\sum_{T\in\mathcal{T}_h} \|h_T(\nabla\cdot\mathbf{E}_h + Xe_h)\|_{0,T}^2\right)^{1/2} \|v\|_1.
\end{aligned}
$$

Now, (4.5) is a direct consequence of (4.7), (4.9), and (4.10). This completes the proof of the lemma. ∎

Now, we present the main result of this paper, which establishes the reliability bound of the estimator.

THEOREM 4.1. *Let* $(u, \boldsymbol{\sigma})$ *satisfy* (2.7) *and let* $(u_h, \boldsymbol{\sigma}_h)$ *satisfy* (3.3). *Let* $\omega \in W^1_\infty(\Omega)$ *be a (weight) function such that* $\omega = 1$ *on* D, $\|\omega\|_{W^1_\infty} \le C$, *and* $0 \le \omega \le 1$. *Then,*

$$\|\nabla e_h\|_{0,D}^2 + \|\mathbf{E}_h\|_{0,D}^2 \le C\,\eta^2.$$

Proof. Since

$$\|\mathbf{E}_h\|_{0,D}^2 \le 2\|\mathbf{E}_h + \nabla e_h\|_{0,D}^2 + 2\|\nabla e_h\|_{0,D}^2 \le 2\sum_{T\in\mathcal{T}_h} \eta_T^2 + 2\|\nabla e_h\|_{0,D}^2,$$

it suffices to show that

$$\|\nabla e_h\|_{0,D}^2 \le C\,\eta^2. \qquad (4.11)$$

To this end, by the definition of ω, we have

$$\|\nabla e_h\|_{0,D}^2 \le \|\omega\,\nabla e_h\|_{0,\Omega}^2 = (\nabla e_h,\, \nabla(\omega^2\, e_h)) - (\omega\nabla e_h, (2\nabla\omega) e_h). \tag{4.12}$$

It follows from integration by parts, the definition of $X\,v = \mathbf{b}\cdot\nabla v + c\,v$, the Cauchy–Schwarz and the triangle inequalities, assumptions on ω, and Lemma 4.1 that

$$\begin{aligned}
&(\nabla e_h, \nabla(\omega^2 e_h))\\
&= (\mathbf{E}_h + \nabla e_h\,,\, \nabla(\omega^2\, e_h)) + (\nabla\cdot\mathbf{E}_h + X\,e_h, \omega^2 e_h) - (X e_h, \omega^2 e_h)\\
&= \Big(\omega\,(\mathbf{E}_h + \nabla\, e_h),\, \omega\nabla e_h + (2\nabla\omega) e_h\Big) + \Big(\nabla\cdot\mathbf{E}_h + X e_h\,, \omega^2\, e_h\Big)\\
&\quad - (\mathbf{b}\cdot\omega\nabla e_h + c\,\omega\, e_h\,, \omega\, e_h)\\
&\le \|\omega\,(\mathbf{E}_h + \nabla e_h)\|_0\Big(\|\omega\nabla e_h\|_0 + C\,\|e_h\|_0\Big) + C\,\eta\,\|\omega^2\, e_h\|_1\\
&\quad + C\,\Big(\|\omega\nabla e_h\|_0 + \|e_h\|_0\Big)\|e_h\|_0.
\end{aligned}$$

By the Poincaré, the triangle and the Cauchy–Schwarz inequalities, and $\|\omega\|_{W^1_\infty} \le C$, we have that

$$\|\omega^2 e_h\|_1 \le C\|\nabla(\omega^2 e_h)\| \le C\Big(\|(2\omega\nabla\omega)e_h\| + \|\omega^2\nabla e_h\|\Big) \le C\Big(\|e_h\|_0 + \|\omega\nabla e_h\|_0\Big)$$

and that

$$(\omega\nabla e_h, (2\nabla\omega)e_h) \le C\,\|\omega\nabla e_h\|_0\,\|e_h\|_0.$$

Since $\|\omega\,(\mathbf{E}_h + \nabla e_h)\|_0 \le \eta$, then combining the above inequalities gives

$$\|\omega\,\nabla e_h\|_{0,\Omega}^2 \le C\Big(\eta\,\|\omega\nabla e_h\|_0 + \eta\,\|e_h\|_0 + \|e_h\|_0\|\omega\nabla e_h\|_0 + \|e_h\|_0^2\Big),$$

which, together with Theorem 3.1, implies

$$\|\omega\,\nabla e_h\|_{0,\Omega}^2 \le C\Big(\eta\,\|\omega\nabla e_h\|_0 + \eta^2\Big).$$

Hence, by the ϵ-inequality, we have

$$\|\omega\nabla e_h\|_{0,\Omega}^2 \le C\,\eta^2,$$

which, together with (4.12), implies the validity of (4.11). This completes the proof of the theorem. ∎

Remark 4.3. For the local efficiency estimate, it is proved in [32] that

$$\eta_T \le C\Big(\|\mathbf{E}_h\|_{0,T}^2 + \|e_h\|_{1,T}^2 + \|h_T(f - f_h)\|_{0,T}^2\Big),$$

where f_h is a polynomial approximation of f and $T \in \mathcal{T}_h$.

5. Numerical experiments. For comparison, this section presents numerical results of an adaptive meshing algorithm using either the goal-oriented a posteriori error estimator defined in (4.2) or the global a posteriori error estimator introduced in [31]:

$$\psi_T = \|\boldsymbol{\sigma}_h + \nabla u_h\|_{0,T}^2 + \|h_T(f - \nabla \cdot \boldsymbol{\sigma}_h - X u_h)\|_{0,T}^2. \tag{5.1}$$

We report their performances for a model problem:

$$\begin{aligned} -\triangle u &= f && \text{in } \Omega = [-1,1] \times [-1,1], \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

discretized by the LS method using piecewise linear conforming finite elements. The exact solution of the test problem is

$$u = (x^2 + y^2)^{0.51}\,(x^2 - 1)(y^2 - 1),$$

and the area of interest is $D = \{(x, y) : (x + 1)^2 + (y + 1)^2 \le \frac{1}{4}\}$. Note that the solution u has low regularity at the origin and that the area of interest D is away from the origin.

The initial mesh is obtained first by partitioning Ω into a regular 10×10 mesh of smaller squares of size $h = 0.2$ and then by dividing each square into two triangles through connecting the lower left corner with the upper right corner. A sequence of meshes is generated by using a standard adaptive meshing algorithm that adopts the Döfler's bulk marking strategy [24]: construct a minimal subset $\mathcal{M}_h$ of $\mathcal{T}_h$ such that

$$\sum_{T \in \mathcal{M}_h} \zeta_T \ge \theta \sum_{T \in \mathcal{T}_h} \zeta_T$$

with $\theta = 0.7$, where $\zeta_T = \psi_T$ or $\zeta_T = \eta_T$. For the mesh refinement, we use the MATLAB function refinemesh, which divides each marked triangle into four congruent triangles and refines some unmarked triangles in order to preserve the triangulation and its quality.

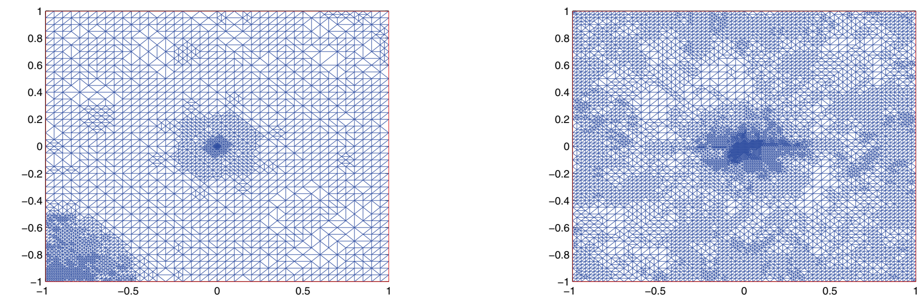
Convergence behavior of the approximate solutions is measured by the following norms:

$$\text{total error} = \|\nabla(u - u_h)\|_{0,D} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,D} \quad \text{and} \quad \text{flux error} = \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,D}.$$

Figure 1 depicts meshes generated using either η_T or ψ_T. Note that our goal-oriented estimator identifies not only the area of interest, but also the source of pollution for mesh refinements. Figure 2 represents convergence behavior of the total error and the flux error from the adaptive meshing algorithm. It shows the superior convergence behavior of the proposed estimator by concentrating computational efforts in the region of interest, and refining outside the region of interest enough to prevent negative influence on the error concerned. Also, we provide convergence behaviors of the errors when one refines only the area of interest in Figure 2 (labeled as "local only"). The pollution effect is observed in this case.

The effectivity index for the proposed estimator is computed by

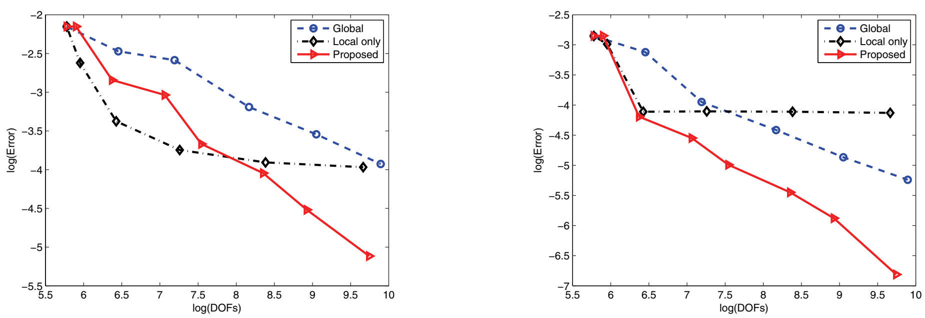
$$\mathcal{I}_P = \frac{\eta}{(\|\nabla(u - u_h)\|_{0,D}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,D}^2)^{1/2}},$$



(a) Proposed, DOFs=17047, $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,D} = 0.0011$

(b) Global, DOFs=19838, $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,D} = 0.0053$

FIG. 1. *Meshes of proposed a posteriori error estimator (left) and global a posteriori error estimator. Area of interest* $D = \{(x, y) : (x + 1)^2 + (y + 1)^2 \leq 1/4\}$.



(a) Convergence behaviors in total error

(b) Convergence behavior in flux error

FIG. 2. *Convergence behaviors.*

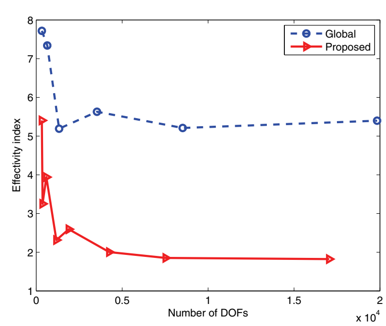


FIG. 3. *Comparison of effectivity index.*

and the effectivity index for the global estimator $\mathcal{I}_G$ is defined similarly. The graphs of the effectivity index $\mathcal{I}_P$ and $\mathcal{I}_G$ are depicted in Figure 3. The effectivity indices for $\mathcal{I}_P$ and $\mathcal{I}_G$ are about 2 and over 5, respectively. Again, our estimator shows superior performance.

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