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A DUAL FINITE ELEMENT METHOD FOR A SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEM*

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Abstract. We present a dual finite element method for a singularly perturbed reaction-diffusion problem. It can be considered a reduced version of the mixed finite element method for approximate solutions. The new method only approximates the dual variables without approximating the primary variable. An approximation for the primary variable is recovered through a simple local L_2 projection. Optimal error estimates for the primary and flux variables are obtained. Our method provides a competitive alternative to other existing numerical methods. For example, our approximate solution for the primary variable does not show a significant numerical oscillation, which is observed in the standard Galerkin methods, and we present a confirming numerical example.

Key words. singularly perturbed, mixed finite element

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1. Introduction.

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a bounded polygonal or smooth domain with Lipschitz-continuous boundary $\partial\Omega$. Consider the singularly perturbed reaction-diffusion problem

$$\begin{cases} -\epsilon^2 \triangle u + b^{-1} u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\epsilon \ll 1$ is a constant. Assume that $f \in L^2(\Omega)$ and that b is a smooth function bounded below and above by positive constants b_0 and b_1, i.e., $0 < b_0 \le b(x) \le b_1$ for almost all $x \in \Omega$. For simplicity, assume that $b_1/b_0 = \mathcal{O}(1)$.

Singularly perturbed problems like (1.1) appear in some applications, and typically their solutions exhibit sharp boundary layers near the boundary (see [33]). Various numerical methods for (1.1) have been introduced and analyzed (see, e.g., [1, 12, 23, 24, 26, 28, 29, 33, 34, 38, 39]). When the solution of (1.1) is sufficiently smooth, it has been demonstrated in [1, 23, 24, 28, 29] that the discretization error in the energy norm,

$$\|v\|_{1,\epsilon} = \left(\epsilon^2 \|\nabla v\|^2_{L_2(\Omega)^n} + \|v\|^2_{L_2(\Omega)}\right)^{1/2} \quad \forall\, v \in H^1_0(\Omega),$$

can be made small uniformly with respect to ϵ when using a proper Shishkin mesh. Nevertheless, for small ϵ, the energy norm is like the L^2 norm, and, hence, it is too weak a norm to measure adequately the discretization error.

Recently, Lin and Stynes in [27] introduced a new finite element method computing both the primal variable u and the dual variables $\boldsymbol{\sigma} = -\nabla u$ simultaneously. The method is proved to be quasi-optimal in a strong norm $(\epsilon^3\|\Delta v\|^2 + \epsilon\|\nabla v\|^2 + \|v\|^2)^{1/2}$.

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Moreover, for a problem posed on the unit square, its error bound on a Shishkin mesh is proved to be uniform in ϵ. This method is based on a new formulation that starts with a first-order system of (1.1) and uses the idea of the least-squares method with a slightly different adjoint operator of the first-order system. In a special case where $b = 1$, the method becomes the least-squares method that is symmetric. In general, their method is asymmetric.

Using the idea of Lin and Stynes [27], Roos and Schopf in [34] developed a C^0 interior penalty method. The method is based on the primary variable and has improved stability properties compared to Galerkin methods on Shishkin meshes. Moreover, balanced (with respect to ϵ) error estimates are obtained. Also, Heuer and Karkulik in [16] developed a discontinuous Petrov–Galerkin method. The method approximates seven unknown variables, including three field variables: primary variable u, the flux $\boldsymbol{\sigma} = -\nabla u$, and $\nabla \cdot \boldsymbol{\sigma}$. They obtained an optimal error estimate in a norm that is balanced in the field variables. However, the resulting algebraic system is large due to simultaneous approximations of all seven variables and, hence, is expensive to solve.

The purpose of this paper is to analyze a two-stage finite element method that has all attractive approximation properties of the Lin and Stynes method. The first stage is to compute the dual variable through a finite element approximation to the dual problem of (1.1) (see (2.4)). The dual problem is symmetric and coercive with respect to a weighted $H(\mathrm{div};\Omega)$ norm $\|\boldsymbol{\tau}\| + \epsilon\|\nabla \cdot \boldsymbol{\tau}\|$ on the dual variable, which, in turn, yields a balanced norm $\|\nabla v\| + \epsilon\|\Delta v\|$ on the primal variable (see [27]). The resulting system of algebraic equations of this stage may be solved by a fast multigrid method (see [2]). The second stage is to compute the primal variable, if needed, through a local L^2 recovery from the computed dual variable. A similar strategy is successfully applied to solve second-order elliptic PDEs in [22].

A standard argument yields that the finite element approximation $\boldsymbol{\sigma}_h$ to the dual problem is optimal with respect to the energy norm (see Theorem 4.1)

$$|||\boldsymbol{\tau}||| = \left(\|\boldsymbol{\tau}\|^2_{L_2(\Omega)} + \epsilon^2\|b^{1/2}\nabla \cdot \boldsymbol{\tau}\|^2_{L_2(\Omega)}\right)^{1/2}.$$

Combining with a result in Lin and Stynes [27], it yields a uniform error estimate on a Shishkin mesh (see Corollary 4.2). The recovered approximation u_h to the primary variable u in the second stage satisfies (see section 3)

$$u_h = P_h u + \epsilon^2 P_h[b\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)], \tag{1.2}$$

where P_h is the local L^2 projection (see (2.7)). Hence, the u_h is a perturbation of the local L^2 projection of the exact solution u and has the following error estimate:

$$\|u - u_h\|_{L_2(\Omega)} \le \|u - P_h u\|_{L_2(\Omega)} + \epsilon\sqrt{b_1} \min_{\boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h^k} |||\boldsymbol{\sigma} - \boldsymbol{\tau}_h|||.$$

In addition to the basic error estimates mentioned above on general meshes, including highly anisotropic meshes, we also present a sharp L^2 error estimate of the dual variable and various local and maximum norm error estimates on a quasi-uniform mesh. In particular, our local maximum norm error estimate shows that the so-called pollution effect is decreasing as $\epsilon \to 0$ (see Theorem 6.5 and Corollary 6.6). The error is bounded by the locally best approximation and weakly depends on the error on a larger domain. Numerically, the recovered primary approximation does not show significant numerical oscillations, which are observed in the standard Galerkin solutions. This is an imperative improvement since the numerical oscillation is artificial and does not exist in the exact solutions. The fact that the approximation u_h is a

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perturbation of the local L^2 projection of the exact solution u provides an explanation for why the u_h does not show significant numerical oscillations.

This paper is organized as follows. In section 2, we introduce our formulation to solve the singularly perturbed problems, and we present numerical methods to approximate the problems in section 3. In section 4, error estimates are presented on general meshes, including highly anisotropic meshes. On quasi-uniform meshes, L_2 and maximum norm error estimates are presented in section 5, and local error estimates are developed in section 6. Numerical experiments are presented in section 7.

2. The dual problem, finite element spaces, and preliminaries. This section introduces the dual problem, the conforming finite element spaces of $H^1(\Omega)$ and $H(\mathrm{div};\Omega)$, and the discrete delta function needed for a priori error estimates in the maximum norm.

2.1. Dual problem. The corresponding minimization problem of (1.1) is to find $u \in H_0^1(\Omega)$ such that

$$J(u) = \inf_{v \in H_0^1(\Omega)} J(v), \tag{2.1}$$

where $J(v)$ is the energy functional given by

$$J(v) = \frac{1}{2}\left(\epsilon^2 \|\nabla v\|^2_{L_2(\Omega)} + \|b^{-1/2} v\|^2_{L_2(\Omega)}\right) - (f,\, v).$$

The dual problem of (2.1) is to find $\boldsymbol{\sigma} \in H(\mathrm{div};\Omega)$ such that

$$J^*(\boldsymbol{\sigma}) = \sup_{\boldsymbol{\tau} \in H(\mathrm{div};\Omega)} J^*(\boldsymbol{\tau}), \tag{2.2}$$

where $H(\mathrm{div};\Omega)$ is the space of all square-integrable vector fields whose divergence is also square-integrable and $J^*(\boldsymbol{\tau})$ is the complementary energy functional given by

$$J^*(\boldsymbol{\tau}) = -\frac{1}{2}\left(\epsilon^2 \|\boldsymbol{\tau}\|^2_{L_2(\Omega)} + \|b^{1/2}(\epsilon^2 \nabla\cdot\boldsymbol{\tau} - f)\|^2_{L_2(\Omega)}\right).$$

By the duality theory (see, e.g., [14]), it is well known that

$$J(u) = J^*(\boldsymbol{\sigma}) \quad \text{and} \quad \boldsymbol{\sigma} = -\nabla u. \tag{2.3}$$

The corresponding variational problem of the maximization problem in (2.2) is to find $\boldsymbol{\sigma} \in H(\mathrm{div};\Omega)$ such that

$$B(\boldsymbol{\sigma},\, \boldsymbol{\tau}) = f(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in H(\mathrm{div};\Omega), \tag{2.4}$$

where the bilinear and linear forms are given by

$$B(\boldsymbol{\sigma},\, \boldsymbol{\tau}) = \epsilon^2 (b\,\nabla\cdot\boldsymbol{\sigma}, \nabla\cdot\boldsymbol{\tau}) + (\boldsymbol{\sigma}, \boldsymbol{\tau}) \quad \text{and} \quad f(\boldsymbol{\tau}) = (b\,f, \nabla\cdot\boldsymbol{\tau}), \tag{2.5}$$

respectively. Denote the induced energy norm of the dual bilinear form by

$$|||\boldsymbol{\tau}||| = B^{1/2}(\boldsymbol{\tau},\, \boldsymbol{\tau}) = \left(\|\boldsymbol{\tau}\|^2_{L_2(\Omega)} + \epsilon^2 \|b^{1/2}\nabla\cdot\boldsymbol{\tau}\|^2_{L_2(\Omega)}\right)^{1/2}. \tag{2.6}$$

This is a balanced norm for the dual variable since $\|\boldsymbol{\tau}\|_{L_2(\Omega)}$ and $\epsilon\,\|\nabla\cdot\boldsymbol{\tau}\|_{L_2(\Omega)}$ have the same scale for problem (1.1) (see [27]).

2.2. Finite element spaces. Let $\mathcal{T}_h$ be a triangulation of the domain Ω consisting of triangles or quadrilaterals. For simplicity, we present our argument in triangular elements. Denote by h_K the diameter of element $K \in \mathcal{T}_h$ and by $h = \max_{K\in\mathcal{T}_h} h_K$ the mesh size of the triangulation $\mathcal{T}_h$.

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Let $P_k(K)$ be the space of polynomials of degree less than or equal to k on an element K. The finite element space for approximating the primal variable is piecewise discontinuous polynomials:

$$V_h^k = \{v \in L^2(\Omega) : v|_K \in P_k(K), \forall K \in \mathcal{T}_h\}.$$

Denote the $H(\mathrm{div};\Omega)$-conforming Raviart–Thomas and Brezzi–Douglas–Marini finite element spaces [7] by

$$RT_k = \{\boldsymbol{\tau} \in H(\mathrm{div};\Omega) : \boldsymbol{\tau}|_K \in P_k(K)^n + \mathbf{x}P_k(K), \forall\, K \in \mathcal{T}_h\}$$

$$\text{and} \quad BDM_k = \{\boldsymbol{\tau} \in H(\mathrm{div};\Omega) : \boldsymbol{\tau}|_K \in P_k(K)^n, \forall\, K \in \mathcal{T}_h\}.$$

The dual variable will be approximated by $\boldsymbol{\Sigma}_h^k = RT_k$ or BDM_{k+1}.

Let $P_h : L^2(\Omega) \to V_h^k$ be the local L^2 projection,

$$(v - P_h v, w) = 0 \quad \forall\, w \in V_h^k, \tag{2.7}$$

and let $\Pi_h : H(\mathrm{div};\Omega) \to \boldsymbol{\Sigma}_h^k$ denote the standard interpolation (Fortin) operator defined in [7]. Then the operators P_h and Π_h satisfy the following commutativity property:

$$\nabla \cdot \Pi_h \boldsymbol{\tau} = P_h \nabla \cdot \boldsymbol{\tau} \quad \forall\, \boldsymbol{\tau} \in H(\mathrm{div};\Omega); \tag{2.8}$$

equivalently, we have

$$\big(\nabla \cdot (\boldsymbol{\tau} - \Pi_h \boldsymbol{\tau}),\, v\big) = \big((I - P_h)\nabla \cdot \boldsymbol{\tau}, v\big) = 0 \quad \forall\, v \in V_h^k. \tag{2.9}$$

2.3. Approximation properties on quasi-uniform meshes. For the local and maximum error estimates in sections 5 and 6, we assume that the underlying meshes are quasi-uniform. Here, we state some of approximation properties that are valid on quasi-uniform meshes.

First, the local L^2 projection operator P_h defined in (2.7) satisfies the following approximation and stability property (see, e.g., [41]): For $1 \le p \le \infty$, $m \ge 0$, and any $T \in \mathcal{T}_h$, there exists a positive constant C such that

$$\|v - P_h v\|_{L_p(T)} \le Ch|v|_{W_p^1(T)} \tag{2.10}$$

and

$$\|P_h v\|_{W_p^m(T)} \le C\|v\|_{W_p^m(T)}. \tag{2.11}$$

Second, the standard interpolation operator $\Pi_h : H(\mathrm{div};\Omega) \to \boldsymbol{\Sigma}_h^k$ satisfies the following approximation property [7]: For each $T \in \mathcal{T}_h$, there exist an integer k and a positive constant C such that for all $1 \le r \le k+1$,

$$\|\boldsymbol{\tau} - \Pi_h \boldsymbol{\tau}\|_{L_2(T)} \le Ch^r |\boldsymbol{\tau}|_{W_2^r(T)} \quad \forall\, \boldsymbol{\tau} \in W_2^r(\Omega). \tag{2.12}$$

Third, let $D \subset \Omega$ be a subdomain and $D_d = \{x \in \Omega : \mathrm{dist}(x, D) < d\}$. Let ω be a sufficiently smooth function. The following superapproximation properties hold [13] for the local L^2 projection operator: For any $1 \le p \le \infty$, there exists a positive constant C such that

$$\|(I - P_h)(\omega v)\|_{L_p(\Omega)} \le Ch\|\omega\|_{W_\infty^{k+1}(\Omega)}\|v\|_{L_p(\Omega)} \quad \forall\, v \in V_h^k. \tag{2.13}$$

Similarly, when $\mathrm{supp}(\omega) \subset D$, $\|\nabla\omega\|_{L_\infty(D)} \le C\, d^{-1}$, and $d > 2h$, the interpolation operator Π_h satisfies the following superapproximation property: For any $1 \le p \le \infty$,

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there exists a positive constant C such that

$$\|(I-\Pi_h)(\omega\boldsymbol{\tau})\|_{L_p(D)^n} \le C\frac{h}{d}\,\|\tau\|_{L_p(D_d)^n} \quad \forall\, \boldsymbol{\tau}\in\boldsymbol{\Sigma}_h^k. \tag{2.14}$$

Also, we need the following inverse inequality [4, 11]: For any $T\in\mathcal{T}_h$ and for $1\le p\le q\le\infty$ and $m\ge 0$, there exists a positive constant C such that

$$\|\boldsymbol{\tau}\|_{[W_q^m(T)]^n} \le Ch^{n/q-n/p-m}\|\boldsymbol{\tau}\|_{[L_p(T)]^n} \quad \forall\, \boldsymbol{\tau}\in\boldsymbol{\Sigma}_h^k. \tag{2.15}$$

In particular, for $m=1$, (2.15) with $p=q=2$ and ∞ implies

$$\begin{cases} \|\boldsymbol{\tau}\|_{W_2^1(T)^n} \le Ch^{-1}\|\boldsymbol{\tau}\|_{L_2(T)^n} & \forall\,\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h^k,\\ \|\boldsymbol{\tau}\|_{W_\infty^1(T)^n} \le Ch^{-1}\|\boldsymbol{\tau}\|_{L_\infty(T)^n} & \forall\,\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h^k. \end{cases} \tag{2.16}$$

2.3.1. Discrete delta function. For the maximum norm error estimates, we need to use the discrete delta function. For any given $x_0\in\Omega$, let $\boldsymbol{\delta}_i^0\in\boldsymbol{\Sigma}_h^k$ be a function such that

$$(\boldsymbol{\tau},\boldsymbol{\delta}_i^0) = [\boldsymbol{\tau}(x_0)]_i \qquad \forall\,\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h^k, \tag{2.17}$$

where $[\boldsymbol{\tau}(x_0)]_i$ is the ith component of the vector $\boldsymbol{\tau}(x_0)$. We need to use the following inequality obtained in [13]: There exists a positive constant C such that

$$\|\boldsymbol{\delta}_i^0\|_{[L_p(\Omega)]^n} + h\|\nabla\cdot\boldsymbol{\delta}_i^0\|_{[L_p(\Omega)]} \le Ch^{n(1/p-1)}. \tag{2.18}$$

In particular, for $p=1$, we have

$$\|\boldsymbol{\delta}_i^0\|_{[L_1(\Omega)]^n} + h\|\nabla\cdot\boldsymbol{\delta}_i^0\|_{[L_1(\Omega)]} \le C. \tag{2.19}$$

3. A two-stage method. This section introduces a two-stage method consisting of (i) computing a finite element approximation to the dual problem in (2.4) and (ii) recovering the primal variable through a local L^2 projection.

Computing the dual variable. The first stage computes a finite element approximation to the dual problem: Find $\boldsymbol{\sigma}_h\in\boldsymbol{\Sigma}_h^k$ such that

$$B(\boldsymbol{\sigma}_h,\,\boldsymbol{\tau}) = f(\boldsymbol{\tau}) \quad \forall\,\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h^k, \tag{3.1}$$

where the bilinear and linear forms are defined in (2.5). The difference between (3.1) and (2.4) gives the following orthogonality:

$$B(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{\tau}) = 0 \quad \forall\,\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h^k. \tag{3.2}$$

Recovering the primal variable. The second stage is to recover the primal variable, if needed, from the computed dual variable $\boldsymbol{\sigma}_h$. Using (1.1) and (2.3), we have

$$u = b\left(f-\epsilon^2\nabla\cdot\boldsymbol{\sigma}\right),$$

which suggests the following recovery:

$$u_h = P_h\left\{b\left(f-\epsilon^2\nabla\cdot\boldsymbol{\sigma}_h\right)\right\}. \tag{3.3}$$

Here P_h is the local L^2 projection defined in (2.7). Using the first equation in (1.1), (3.3) becomes

$$u_h = P_h u + \epsilon^2 P_h\left\{b\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\right\}. \tag{3.4}$$

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It follows from (3.3), the definition of P_h, and the equation in (3.1) that for any $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h^k$,

$$(u_h, \nabla \cdot \boldsymbol{\tau}) = (P_h[b(f - \epsilon^2 \nabla \cdot \boldsymbol{\sigma}_h)], \nabla \cdot \boldsymbol{\tau}) = (b(f - \epsilon^2 \nabla \cdot \boldsymbol{\sigma}_h), \nabla \cdot \boldsymbol{\tau}) = (\boldsymbol{\sigma}_h, \boldsymbol{\tau}).$$

Hence, the recovery of the primal variable is equivalent to find $u_h \in V_h^k$ such that

$$(u_h, \nabla \cdot \boldsymbol{\tau}) = (\boldsymbol{\sigma}_h, \boldsymbol{\tau}), \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h^k. \tag{3.5}$$

4. Error estimates on general meshes. For the numerical method defined in (3.1) and (3.3), this section presents a priori error estimates in the L^2 norm on general meshes including highly anisotropic meshes.

First, using (3.2) and the definition of $|||\cdot|||$ in (2.6), the following basic error estimate in the energy norm is obtained by a standard argument (e.g., see [10]).

THEOREM 4.1. *Let $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h^k$ be the solutions of* (2.4) *and* (3.1)*, respectively. Then*

$$|||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||| \le \min_{\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h^k} |||\boldsymbol{\sigma} - \boldsymbol{\tau}|||.$$

Error estimate on a Shishkin mesh. For problem (1.1) on the unit square with sufficiently smooth f and b, Lin and Stynes [27] constructs a Shishkin mesh that is piecewise equidistant with N mesh intervals in each coordinate direction. Let $(\boldsymbol{\sigma}_h^{LS}\, u_h^{LS}) \in RT_0 \times S_h^1$ be the finite element approximation obtained from their approach, where RT_0 is the lowest-order Raviart–Thomas element on rectangular meshes and S_h^1 is the continuous bilinear element. They proved the following estimate

$$\epsilon^{3/2} \|\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^{LS})\|_{L_2(\Omega)} + \epsilon^{1/2} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^{LS}\|_{L_2(\Omega)} \le C N^{-1} \ln N, \tag{4.1}$$

where the constant C is independent of ϵ.

Combining Theorem 4.1 and (4.1), the same uniform error estimate with respect to the ϵ is valid for the finite element approximation $\boldsymbol{\sigma}_h$.

COROLLARY 4.2. *Let $\boldsymbol{\sigma}$ be the solution of* (2.4) *with $\Omega = (0, 1)^2$. Let $\boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h^0$ be the solution of* (3.1) *on the Shishkin mesh constructed in* [27]*. Then there exists a positive constant C independent of ϵ such that*

$$\epsilon^{3/2} \|\sqrt{b} \nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L_2(\Omega)} + \epsilon^{1/2} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(\Omega)} \le C N^{-1} \ln N. \tag{4.2}$$

Proof. Theorem 4.1 with $\boldsymbol{\tau} = \boldsymbol{\sigma}^{LS}$ and the fact that $b(x) \le b_1$ give

$$\begin{aligned}
&\epsilon^{1/2} \left(\epsilon \|\sqrt{b} \nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L_2(\Omega)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(\Omega)} \right) \\
&\le \epsilon^{1/2} \left(\epsilon \|\sqrt{b} \nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^{LS})\|_{L_2(\Omega)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^{LS}\|_{L_2(\Omega)} \right) \\
&\le \max\{1, \sqrt{b_1}\} \epsilon^{1/2} \left(\epsilon \|\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^{LS})\|_{L_2(\Omega)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^{LS}\|_{L_2(\Omega)} \right),
\end{aligned}$$

which, together with (4.1), implies (4.2). ∎

Next, we establish a superconvergence result on $\|P_h u - u_h\|_{L_2(\Omega)}$. This estimate provides a sharp L^2 norm error estimate for the error $\|u - u_h\|_{L_2(\Omega)}$.

THEOREM 4.3. *Let u and u_h be the solutions of* (1.1) *and* (3.5)*, respectively. For the local L^2 projection operator P_h, we have*

$$\|P_h u - u_h\|_{L_2(\Omega)} \le \epsilon \sqrt{b_1} \min_{\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h^k} |||\boldsymbol{\sigma} - \boldsymbol{\tau}|||. \tag{4.3}$$

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Proof. For any $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h^k$, the definition of P_h and integration by parts gives

$$(P_h u, \nabla \cdot \boldsymbol{\tau}) = (u, \nabla \cdot \boldsymbol{\tau}) = (\boldsymbol{\sigma}, \boldsymbol{\tau}),$$

which, together with (3.5), yields

$$(P_h u - u_h\,,\, \nabla \cdot \boldsymbol{\tau}) = (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h^k. \tag{4.4}$$

To bound $\|P_h u - u_h\|_{L_2(\Omega)}$, consider the auxiliary problem

$$\begin{cases} \boldsymbol{\eta} + \nabla \lambda = 0, & \text{in} \quad \Omega, \\ \nabla \cdot \boldsymbol{\eta} = P_h u - u_h, & \text{in} \quad \Omega \end{cases} \tag{4.5}$$

with homogeneous boundary condition $\lambda = 0$ on $\partial\Omega$. It follows from (4.5); the definition of P_h; the commutativity property in (2.8), (4.4), and (3.2); and the Cauchy–Schwarz inequality that

$$\begin{aligned} \|P_h u - u_h\|_{L_2(\Omega)}^2 &= (P_h u - u_h, \nabla \cdot \boldsymbol{\eta}) = (P_h u - u_h, P_h \nabla \cdot \boldsymbol{\eta}) = (P_h u - u_h, \nabla \cdot \Pi_h \boldsymbol{\eta}) \\ &= (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \Pi_h \boldsymbol{\eta}) = -\epsilon^2 (b \nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), \nabla \cdot (\Pi_h \boldsymbol{\eta})) \\ &= -\epsilon^2 (b \nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), P_h u - u_h) \\ &\le b_1^{1/2} \epsilon^2 \|b^{1/2} \nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L_2(\Omega)} \|P_h u - u_h\|_{L_2(\Omega)}, \end{aligned} \tag{4.6}$$

which yields

$$\|P_h u - u_h\|_{L_2(\Omega)} \le \epsilon^2 \sqrt{b_1} \|b^{1/2} \nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L_2(\Omega)}.$$

Now (4.3) is a direct consequence of Theorem 4.1. This completes the proof of the theorem. ∎

THEOREM 4.4. *Let u and u_h be the solutions of* (1.1) *and* (3.5)*, respectively. Then*

$$\|u - u_h\|_{L_2(\Omega)} \le \|u - P_h u\|_{L_2(\Omega)} + \epsilon \sqrt{b_1} \min_{\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h^k} |\|\boldsymbol{\sigma} - \boldsymbol{\tau}\||. \tag{4.7}$$

Proof. Equation (4.7) follows directly from the triangle inequality and Theorem 4.3. ∎

5. Error estimates on quasi-uniform meshes. This section provides error estimates in the L^2 and maximum norms on a quasi-uniform triangulation $\mathcal{T}_h$.

Theorem 4.1 implies the following L^2 norm error estimate for the dual variable:

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(\Omega)} \le \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L_2(\Omega)} + \epsilon \|\sqrt{b} \nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_2(\Omega)}.$$

By using the superapproximation property in (2.13), the dependence on the L^2 norm of the divergence of the interpolation error may be further weakened.

The following equality is an immediate consequence of (2.8):

$$\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}) = \nabla \cdot \boldsymbol{\sigma} - P_h \nabla \cdot \boldsymbol{\sigma}. \tag{5.1}$$

THEOREM 5.1. *Let $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h^k$ be the solutions of* (2.4) *and* (3.1)*, respectively. Then there exists a positive constant C that is independent of ϵ but that may depend on b such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(\Omega)} \le 2\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L_2(\Omega)} + C\epsilon^2 \|\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_2(\Omega)}. \tag{5.2}$$

In the case that $b = 1$, we have

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(\Omega)} \le \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L_2(\Omega)}. \tag{5.3}$$

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Proof. By the assumption of the theorem, the superapproximation property in (2.13) with $v = \nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)$ and $\omega = b$, and the inverse inequality (2.16), we have

$$\begin{aligned} &\|(I - P_h)[b\nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)]\|_{L_2(\Omega)} \\ &\le Ch\|\nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L_2(\Omega)} \le C\|\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(\Omega)}. \end{aligned} \tag{5.4}$$

It follows from the orthogonality in (3.2), (5.1), the property of P_h in (2.7), the Cauchy–Schwarz inequality, (5.4), and the triangle inequality that

$$\begin{aligned} &(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) = -\epsilon^2 \left(b\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), \nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\right) \\ &= -\epsilon^2 \left(b\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}), \nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\right) - \epsilon^2 \|b^{1/2}\nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|^2_{L_2(\Omega)} \\ &\le -\epsilon^2 \left(\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}), b\nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\right) \\ &\le -\epsilon^2 \left(\nabla \cdot \boldsymbol{\sigma} - P_h \nabla \cdot \boldsymbol{\sigma}, b\nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\right) \\ &= -\epsilon^2 \big(\nabla \cdot \boldsymbol{\sigma} - P_h \nabla \cdot \boldsymbol{\sigma}, b\nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) - P_h\{b\nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\}\big) \\ &= -\epsilon^2 \big(\nabla \cdot \boldsymbol{\sigma} - P_h \nabla \cdot \boldsymbol{\sigma}, (I - P_h)[b\nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h]\big) \\ &\le \epsilon^2 \|\nabla \cdot \boldsymbol{\sigma} - P_h \nabla \cdot \boldsymbol{\sigma}\|_{L_2(\Omega)} \|(I - P_h)[b\nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h]\|_{L_2(\Omega)} \\ &\le C\epsilon^2 \|\nabla \cdot \boldsymbol{\sigma} - P_h \nabla \cdot \boldsymbol{\sigma}\|_{L_2(\Omega)} \|\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(\Omega)} \\ &\le C\epsilon^2 \|\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_2(\Omega)} \big(\|\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}\|_{L_2(\Omega)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(\Omega)}\big) \end{aligned} \tag{5.5}$$

By the Cauchy–Schwarz inequality and (5.5), we have

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{L_2(\Omega)} &= (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \\ &\le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(\Omega)} \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L_2(\Omega)} + (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \\ &\le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(\Omega)} \big(\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L_2(\Omega)} + C\epsilon^2 \|\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_2(\Omega)}\big) \\ &\quad + C\epsilon^2 \|\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_2(\Omega)} \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L_2(\Omega)}. \end{aligned} \tag{5.6}$$

Now the validity of (5.2) is a direct consequence of the arithmetic-geometric inequality.

When $b = 1$, the second equality in (5.5) and (2.9) give

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) = -\epsilon^2 \|\nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|^2_{L_2(\Omega)} \le 0.$$

Together with the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{L_2(\Omega)} &= (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \le (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}) \\ &\le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L_2(\Omega)]^n} \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{[L_2(\Omega)]^n}, \end{aligned}$$

which implies the validity of (5.3). This completes the proof of the theorem. ∎

We also present maximum norm error estimates first for the dual variable and then for the primary variable. We use the discrete delta function defined in (2.17) for our estimates for the dual variable.

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THEOREM 5.2. *Let $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h^k$ be the solutions of* (2.4) *and* (3.1)*, respectively. Assume that $\epsilon < h^{1+\delta}$ for some $\delta > 0$. Then there exists $h_0 > 0$ such that for any $h < h_0$, there exists a positive constant C that is independent of ϵ but that may depend on b such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_\infty(\Omega)^n} \le C\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L_\infty(\Omega)^n} + C\,\epsilon^2\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|_{L_\infty(\Omega)}. \tag{5.7}$$

Proof. To show the validity of (5.7), by the triangle inequality, it suffices to prove that

$$\|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_\infty(\Omega)^n} \le C\left(\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L_\infty(\Omega)^n} + \epsilon^2\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|_{L_\infty(\Omega)}\right). \tag{5.8}$$

Assume that $\mathbf{x}_0$ is a point such that $\|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_\infty(\Omega)^n} = \left|[(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)(\mathbf{x}_0)]_i\right|$. The discrete delta function and the orthogonality in (3.2) give

$$\begin{aligned}\|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_\infty(\Omega)^n} &= (\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\,,\,\boldsymbol{\delta}_i^0) = (\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}\,,\,\boldsymbol{\delta}_i^0) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\,,\,\boldsymbol{\delta}_i^0)\\ &= \left(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}\,,\,\boldsymbol{\delta}_i^0\right) - \epsilon^2\left(b\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\,,\,\nabla\cdot\boldsymbol{\delta}_i^0\right).\end{aligned} \tag{5.9}$$

Using (5.1) and the orthogonality of the operator P_h in (2.7), we have

$$\begin{aligned}&\left(b\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\,,\,\nabla\cdot\boldsymbol{\delta}_0^i\right) = \left((I - P_h)\nabla\cdot\boldsymbol{\sigma}\,,\,b\nabla\cdot\boldsymbol{\delta}_0^i\right)\\ &= \left((I - P_h)\nabla\cdot\boldsymbol{\sigma}\,,\,(I - P_h)(b\nabla\cdot\boldsymbol{\delta}_0^i)\right) = \left(\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\,,\,(I - P_h)(b\nabla\cdot\boldsymbol{\delta}_0^i)\right),\end{aligned}$$

which, together with (5.9), Hölder's inequality, the superapproximation property in (2.13) with $\omega = b$ and $v = \nabla\cdot\boldsymbol{\delta}_0^i$, and the inverse inequality (2.16), yields

$$\begin{aligned}&\|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_\infty(\Omega)^n}\\ &= \left(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}\,,\,\boldsymbol{\delta}_i^0\right) - \epsilon^2\left(\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}), (I - P_h)(b\nabla\cdot\boldsymbol{\delta}_0^i)\right)\\ &\quad - \epsilon^2\left(b\nabla\cdot(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), \nabla\cdot\boldsymbol{\delta}_i^0\right)\\ &\le \|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L_\infty(\Omega)^n}\|\boldsymbol{\delta}_i^0\|_{L_1(\Omega)} + \epsilon^2\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|_{L_\infty(\Omega)^n}\|(I - P_h)(b\nabla\cdot\boldsymbol{\delta}_i^0)\|_{L_1(\Omega)}\\ &\quad + C\epsilon^2\|\nabla\cdot(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L_\infty(\Omega)^n}\|\nabla\cdot\boldsymbol{\delta}_i^0\|_{L_1(\Omega)}\\ &\le C\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L_\infty(\Omega)^n} + C\epsilon^2 h\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|_{L_\infty(\Omega)^n}\|\nabla\cdot\boldsymbol{\delta}_i^0\|_{L_1(\Omega)}\\ &\quad + C(\epsilon^2/h)\|\nabla\cdot(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L_\infty(\Omega)^n}\\ &\le C\left(\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L_\infty(\Omega)^n} + \epsilon^2\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|_{L_\infty(\Omega)^n}\right) + C\frac{\epsilon^2}{h^2}\|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_\infty(\Omega)^n}.\end{aligned}$$

For sufficiently small h, we have that $C\frac{\epsilon^2}{h^2} < Ch^{2\delta} \le \frac{1}{2}$, which, together with the above inequality, shows the validity of (5.8) and, hence, the theorem. This completes the proof of the theorem. ∎

THEOREM 5.3. *Let u and u_h be the solutions of* (1.1) *and* (3.5)*, respectively. Let P_h be the local L_2 projection defined in* (2.7) *and u_h be the approximate solution defined in* (3.3)*. Then, under the same assumptions as in Theorem* 5.2*,*

$$\begin{aligned}\|u - u_h\|_{L_\infty(\Omega)} &\le \|u - P_h u\|_{L_\infty(\Omega)} + C\,\epsilon\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L_\infty(\Omega)^n}\\ &\quad + C\,\epsilon^2\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|_{L_\infty(\Omega)}.\end{aligned} \tag{5.10}$$

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Proof. By the definition of u_h given in (3.3), we have

$$u - u_h = (u - P_h u) + (P_h u - u_h) = (u - P_h u) + \epsilon^2 P_h\big[b\nabla\cdot(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\big],$$

which, together with (5.11), the triangle inequality, the boundedness of b, and (2.11), gives

$$\|u - u_h\|_{L_\infty(\Omega)} \le \|u - P_h u\|_{L_\infty(\Omega)} + C\epsilon^2\|\nabla\cdot(\boldsymbol{\sigma}_h - \boldsymbol{\sigma})\|_{L_\infty(\Omega)}. \tag{5.11}$$

Using the triangle and inverse inequalities and Theorem 5.2, we have

$$\begin{aligned}
&\|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L_\infty(\Omega)} \le \|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}_h)\|_{L_\infty(\Omega)} + \|\nabla\cdot(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L_\infty(\Omega)}\\
&\le \|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}_h)\|_{L_\infty(\Omega)} + Ch^{-1}\|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_\infty(\Omega)^n}\\
&\le \|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}_h)\|_{L_\infty(\Omega)} + Ch^{-1}\big(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_\infty(\Omega)^n} + \|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L_\infty(\Omega)^n}\big)\\
&\le \|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|_{L_\infty(\Omega)} + \frac{C}{h}\big(\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L_\infty(\Omega)^n} + \epsilon^2\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|_{L_\infty(\Omega)}\big)\\
&\le C\left(\left(1 + \frac{\epsilon^2}{h}\right)\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|_{L_\infty(\Omega)} + h^{-1}\,\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L_\infty(\Omega)^n}\right)\\
&\le C\left(\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|_{L_\infty(\Omega)} + h^{-1}\,\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L_\infty(\Omega)^n}\right),
\end{aligned}$$

which, together with (5.11) and the fact that $\epsilon < h^{1+\delta}$, implies (5.10). This completes the proof of the theorem. ∎

6. Local error estimates. In this section, we obtain error estimates on a subset $D \subset \Omega$ on a quasi-uniform triangulation $\mathcal{T}_h$. For $d > 0$, let

$$D_d = \{x \in \Omega : \mathrm{dist}(x, D) < d\}.$$

Throughout this section, let $0 \le \omega(x) \le 1$ be a cutoff function such that $\omega(x) = 1$ for $x \in D$ and $\omega(x) = 0$ for $x \in D_d^c$ and that $\|\nabla\omega\|_{L_\infty(D_d)} \le \frac{C}{d}$.

Also, we use the following approximation property for the coefficient b: Let b_I denote the piecewise constant on the quasi-uniform mesh $\mathcal{T}_h$ satisfying

$$\|b - b_I\|_{L_\infty(T)} \le Ch_T\|b\|_{W^1_\infty(T)} \le Ch_T \tag{6.1}$$

for any $T \in \mathcal{T}_h$; see [4].

Remark 6.1. The b_I can be considered as a local L_2 projection onto the space of piecewise constants on $\mathcal{T}_h$. When b is nonnegative, b_I is also nonnegative.

LEMMA 6.1. *Let $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_h$ be the solutions of* (2.4) *and* (3.1)*, respectively. Assume that $\epsilon < h < d$; then we have*

$$\big(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h,\, \Pi_h(\omega^2(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h))\big) \le C\Big(\epsilon^2\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|^2_{L_2(D_d)} + \frac{h}{d}\|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{L_2(D_d)}\Big).$$

Proof. First, using the orthogonality property (3.2), we have

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \Pi_h(\omega^2(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h))) = -\epsilon^2(b\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h),\, \nabla\cdot\Pi_h(\omega^2(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h))). \tag{6.2}$$

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Observe that using $\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\in V_h^k$ and b_I in (6.1) being nonnegative piecewise constant, we have $b_I\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\in V_h^k$. Using this and (2.7), we have

$$
\begin{aligned}
&-(b_I\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\, P_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)))\\
&\qquad=-(b_I\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\, (\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)))\\
&\qquad=-(b_I(\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h))^2,\, \omega^2)\le 0. \qquad (6.3)
\end{aligned}
$$

Using the above inequality, (6.1), $\operatorname{supp}(\omega)\subset D_d$, the Cauchy–Schwarz inequality, (2.11), and (2.16), we have

$$
\begin{aligned}
&-(b\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\, P_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)))\\
&=-((b-b_I)\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\, P_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)))\\
&\qquad-(b_I\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\, P_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)))\\
&\le-((b-b_I)\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\, P_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)))\\
&\le\|b-b_I\|_{L_\infty(D_d)}\|\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{L_2(D_d)}\|P_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{L_2(D_d)}\\
&\le Ch\|\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{L_2(D_d)}\|\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{L_2(D_d)}\\
&\le C\|\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L_2(D_d)}\|\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{L_2(D_d)}. \qquad (6.4)
\end{aligned}
$$

Now, using (2.8), (6.4), the Cauchy–Schwarz inequality, and $\|\nabla\omega\|_{L_\infty}\le\frac{C}{d}$, we have

$$
\begin{aligned}
&-\epsilon^2\Big(b\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\, \nabla\cdot\Pi_h(\omega^2(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\Big)\\
&=-\epsilon^2\Big(b\nabla\cdot(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}),\, \nabla\cdot\Pi_h(\omega^2(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\Big)\\
&\qquad-\epsilon^2\Big(b\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\, \nabla\cdot\Pi_h(\omega^2(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\Big)\\
&=-\epsilon^2\Big(b\nabla\cdot(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}),\, P_h\nabla\cdot(\omega^2(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\Big)\\
&\qquad-\epsilon^2\Big(b\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\, P_h\nabla\cdot(\omega^2(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h))\Big)\\
&=-\epsilon^2\Big(b\nabla\cdot(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}),\, P_h(2\omega\nabla\omega\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)+\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h))\Big)\\
&\quad-\epsilon^2\Big(b\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\, P_h(2\omega\nabla\omega\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)+\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h))\Big)\\
&\le-\epsilon^2\Big(b\nabla\cdot(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}),\, P_h(2\omega\nabla\omega\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)+\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h))\Big)\\
&\qquad-\epsilon^2\Big(b\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h),\, P_h(2\omega\nabla\omega\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h))\Big)\\
&\qquad+C\epsilon^2\|\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L_2(D_d))}\|\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{L_2(D_d))}\\
&\le C\frac{\epsilon^2}{d}\|\nabla\cdot(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma})\|_{L_2(D_d)}\|\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L_2(D_d)}\\
&\quad-\epsilon^2(b\nabla\cdot(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}),\, P_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)))\\
&\qquad+C\frac{\epsilon^2}{d}\|\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{L_2(D_d)}\|\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L_2(D_d)}\\
&=I_1+I_2+I_3. \qquad (6.5)
\end{aligned}
$$

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For I_1, using $ab \le a^2 + b^2$ and $\epsilon < h < d$, we have

$$\begin{aligned} I_1 &= C\frac{\epsilon^2}{h}\frac{h}{d}\|\nabla\cdot(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma})\|_{L_2(D_d)}\|\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L_2(D_d)} \\ &\le C\frac{\epsilon^4}{h^2}\|\nabla\cdot(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma})\|^2_{L_2(D_d)} + C\frac{h^2}{d^2}\|\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|^2_{L_2(D_d)} \\ &\le C\epsilon^2\|\nabla\cdot(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma})\|^2_{L_2(D_d)} + C\frac{h}{d}\|\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|^2_{L_2(D_d)}. \end{aligned} \tag{6.6}$$

For I_2, let b_I be the piecewise constant in (6.1). Then $b_I P_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)) \in V_h^k$. Hence, using (2.7), we obtain

$$(\nabla\cdot\boldsymbol{\sigma} - P_h\nabla\cdot\boldsymbol{\sigma}\,,\, b_I P_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h))) = 0. \tag{6.7}$$

Also, using (6.1), (2.11), $0 \le \omega \le 1$, and (2.16), we have

$$\begin{aligned} &\|(b-b_I)P_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h))\|_{L_2(D_d)} \\ &\le \|b-b_I\|_{L_\infty(D_d)}\|P_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h))\|_{L_2(D_d)} \\ &\le Ch\|P_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h))\|_{L_2(D_d)} \le Ch\|\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{L_2(D_d)} \\ &\le Ch\|\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{L_2(D_d)} \\ &\le C\|\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L_2(D_d)}. \end{aligned} \tag{6.8}$$

Now, using (5.1), (6.7), $\mathrm{supp}(\omega) \subset D_d$, the Cauchy–Schwarz inequality, (6.8), and $\epsilon < h < d < 1$, we have

$$\begin{aligned} I_2 &= -\epsilon^2(b\nabla\cdot(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}),\, P_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h))) \\ &= -\epsilon^2(\nabla\cdot(\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}),\, bP_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h))) \\ &= -\epsilon^2(\nabla\cdot\boldsymbol{\sigma}-P_h\nabla\cdot\boldsymbol{\sigma},\, bP_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h))) \\ &= -\epsilon^2\Big(\nabla\cdot\boldsymbol{\sigma}-P_h\nabla\cdot\boldsymbol{\sigma},\, bP_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)) - b_I P_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h))\Big) \\ &= -\epsilon^2\Big(\nabla\cdot\boldsymbol{\sigma}-P_h\nabla\cdot\boldsymbol{\sigma},\, (b-b_I)P_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h))\Big) \\ &\le \epsilon^2\|\nabla\cdot\boldsymbol{\sigma}-P_h\nabla\cdot\boldsymbol{\sigma}\|_{L_2(D_d)}\|(b-b_I)P_h(\omega^2\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h))\|_{L_2(D_d)} \\ &\le C\epsilon^2\|\nabla\cdot\boldsymbol{\sigma}-P_h\nabla\cdot\boldsymbol{\sigma}\|_{L_2(D_d)}\|\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L_2(D_d)} \\ &\le C\epsilon^2\|\nabla\cdot\boldsymbol{\sigma}-P_h\nabla\cdot\boldsymbol{\sigma}\|^2_{L_2(D_d)} + C\epsilon^2\|\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|^2_{L_2(D_d)} \\ &\le C\epsilon^2\|\nabla\cdot\boldsymbol{\sigma}-P_h\nabla\cdot\boldsymbol{\sigma}\|^2_{L_2(D_d)} + C\frac{h}{d}\|\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|^2_{L_2(D_d)}. \end{aligned} \tag{6.9}$$

For I_3, using the inverse inequality (2.16) and $\epsilon < h$, we have

$$\begin{aligned} I_3 &= C\frac{\epsilon^2}{d}\|\nabla\cdot(\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{L_2(D_d)}\|\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L_2(D_d)} \\ &\le C\frac{\epsilon^2}{hd}\|\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L_2(D_d)}\|\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L_2(D_d)} \\ &\le C\frac{\epsilon}{d}\|\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|^2_{L_2(D_d)} \le C\frac{h}{d}\|\Pi_h\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|^2_{L_2(D_d)} \end{aligned} \tag{6.10}$$

Plugging (6.6), (6.9), and (6.10) into (6.5) and then the resulting inequality into (6.2), we obtain the desired inequality. This completes the proof. ∎

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THEOREM 6.2. *Let $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_h$ be the solutions of* (2.4) *and* (3.1)*, respectively. Assume that $\epsilon < h < d$; then we have*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(D)} \le C\left(\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L_2(D_d)} + \frac{h}{d}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(D_d)} + \epsilon\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|_{L_2(D_d)}\right).$$

Proof. It follows from the Cauchy–Schwarz inequality, (2.14), Lemma 6.1, Young's inequality, and the triangle inequality that

$$\begin{aligned}
\|\omega(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|^2_{L_2(D_d)} &= \big(\omega^2(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h),\, \boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\big) \\
&\quad + \big(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h,\, (I - \Pi_h)(\omega^2(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h))\big) + \big(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h,\, \Pi_h(\omega^2(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h))\big) \\
&\le \|\omega(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L_2(D_d)}\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L_2(D_d)} + C\frac{h}{d}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(D_d)}\|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(D_d)} \\
&\quad + C\left(\epsilon^2\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|^2_{L_2(D_d)} + \frac{h}{d}\|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{L_2(D_d)}\right) \\
&\le \frac{1}{2}\|\omega(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|^2_{L_2(D_d)} + \frac{1}{2}\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|^2_{L_2(D_d)} \\
&\quad + C\frac{h}{d}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(D_d)}\Big(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(D_d)} + \|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L_2(D_d)}\Big) \\
&\quad + C\left(\epsilon^2\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|^2_{L_2(D_d)} + \frac{2h}{d}\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|^2_{L_2(D_d)} + \frac{2h}{d}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{L_2(D_d)}\right).
\end{aligned}$$

Now, isolating $\|\omega(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L_2(D_d)}$ in the above equation and using $h < d$ and Young's inequality, we have

$$\begin{aligned}
\|\omega(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|^2_{L_2(D_d)} &\le C\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|^2_{L_2(D_d)} + C\epsilon^2\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|^2_{L_2(D_d)} \\
&\quad + C\frac{h}{d}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{L_2(D_d)} + C\frac{h}{d}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(D_d)}\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{L_2(D_d)} \\
&\le C\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|^2_{L_2(D_d)} + C\epsilon^2\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|^2_{L_2(D_d)} + C\frac{h}{d}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{L_2(D_d)},
\end{aligned}$$

which implies

$$\begin{aligned}
\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{L_2(D)} &\le \|\omega(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|^2_{L_2(D_d)} \\
&\le C\left(\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|^2_{L_2(D_d)} + \frac{h}{d}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{L_2(D_d)} + \epsilon^2\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|^2_{L_2(D_d)}\right).
\end{aligned}$$

Applying the above inequality to $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{0,D_d}$, we have

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{L_2(D)} \le C\left(\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|^2_{L_2(D_{2d})} + \frac{h^2}{d^2}\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{L_2(D_{2d})} + \epsilon^2\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|^2_{L_2(D_{2d})}\right).$$

Now, scaling back from $2d$ to d, we obtain the desired inequality. This completes the proof of the theorem. ∎

Now, by repeated application of the above theorem, we obtain the following result.

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COROLLARY 6.3. *Let $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_h$ be the solutions of* (2.4) *and* (3.1)*, respectively. Assume that $\epsilon < h < d^{1/(1-\delta)}$, i.e., $d > h^{1-\delta}$, for some $\delta > 0$. Then there exists $h_0 > 0$ such that for any $h < h_0$,*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(D)} \le C \cdot \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L_2(D_{sd})} + C^s \left(\frac{h}{d}\right)^s \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(D_{sd})} + C\epsilon \|\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_2(D_{sd})}, \text{ for } s = 1, 2, \ldots.$$

Proof. First, let C be the constant in Theorem 6.2, and let h_0 satisfy

$$Ch_0^\delta < \frac{1}{2}.$$

Then, for any $h < h_0$, using $d > h^{1-\delta}$, we have

$$C\frac{h}{d} \le Ch^\delta \le Ch_0^\delta < \frac{1}{2}.$$

Now, applying Theorem 6.2 repeatedly for $D, D_d, \ldots, D_{(s-1)d}$, we obtain

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(D)} \le 2C(s)\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L_2(D_{sd})} + C^s \left(\frac{h}{d}\right)^s \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(D_{sd})} + C(s)\epsilon \|\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_2(D_{sd})},$$

where

$$\begin{aligned} C(s) &= C\left(1 + \left(C\frac{h}{d}\right) + \left(C\frac{h}{d}\right)^2 + \cdots + \left(C\frac{h}{d}\right)^{s-1}\right) \\ &= C\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{s-1}\right) < 2C. \end{aligned}$$

This completes the proof. ∎

Next, we prove local L_2 norm error estimates for $u - u_h$. Note that u_h is defined using the simple L_2 projection operator P_h and computed solution $\boldsymbol{\sigma}_h$, i.e.,

$$u_h = P_h\{b(f - \epsilon^2 \nabla \cdot \boldsymbol{\sigma}_h)\} = P_h u + \epsilon^2 P_h(b\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)). \tag{6.11}$$

THEOREM 6.4. *Let u_h be the approximate solution defined in* (3.3)*. Then*

$$\begin{aligned} \|u - u_h\|_{L_2(D)} &\le \|u - P_h u\|_{L_2(D)} + C\epsilon^2 \|\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_2(D_d)} \\ &\quad + C\frac{\epsilon^2}{h}\Big(\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L_2(D_d)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(D_d)}\Big). \end{aligned} \tag{6.12}$$

Proof. Using (6.11), the triangle inequality, and the inverse inequality (2.16), we have

$$\begin{aligned} &\|u - u_h\|_{L_2(D)} \\ &\quad = \|u - P_h u\|_{L_2(D)} + \epsilon^2 \|P_h\{b\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\}\|_{L_2(D)} \\ &\quad \le \|u - P_h u\|_{L_2(D)} + C\epsilon^2 \|\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L_2(D)} \\ &\quad \le \|u - P_h u\|_{L_2(D)} + C\epsilon^2 \|\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_2(D)} + C\epsilon^2 \|\nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma})_h\|_{L_2(D)} \\ &\quad \le \|u - P_h u\|_{L_2(D)} + C\epsilon^2 \|\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_2(D)} + C\frac{\epsilon^2}{h}\|\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(D_d)} \\ &\quad \le \|u - P_h u\|_{L_2(D)} + C\epsilon^2 \|\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_2(D_d)} \\ &\quad + C\frac{\epsilon^2}{h}(\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L_2(D_d)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(D_d)}). \end{aligned} \tag{6.13}$$

This completes the proof. ∎

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We prove the following maximum norm error estimate for $u - u_h$ on a single element $T \in \mathcal{T}_h$ for $\Omega \subset \mathbb{R}^2$ using Corollary 6.3. Note that T can be replaced by a region $D \subset \Omega$. Roughly speaking, the estimates show that the error is bounded by locally best approximation and weak dependence on the error on larger domains.

THEOREM 6.5. *Let u_h be the approximate solution defined in* (3.3) *and $n = 2$. Then, under the same assumptions as in Corollary* 6.3*, we have*

$$\begin{aligned}\|u - u_h\|_{L_\infty(T)} &\le C(\|u - P_h u\|_{L_\infty(T)} + \epsilon^2 \|\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_\infty(T)}) \\ &\quad + C\left(\frac{\epsilon}{h}\right)^2 \Bigg(\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L_2(T_{sd})} + C^s \left(\frac{h}{d}\right)^s \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(T_{sd})} \\ &\quad + \epsilon \|\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_2(T_{sd})} \Bigg)\end{aligned} \tag{6.14}$$

for $s = 1, 2, \ldots$.

Remark 6.2. Our local maximum norm error estimate shows similar convergence behavior to the one presented in [35, Theorem 7] in the sense that the local error is bounded by locally best approximation and weak dependence on larger domains.

Proof. Using (6.11), the triangle inequality, the boundedness of b, the inverse inequality (2.16), and (2.15), we have

$$\begin{aligned}\|u - u_h\|_{L_\infty(T)} &= \|u - P_h u\|_{L_\infty(T)} + \epsilon^2 \|P_h\{b \nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\}\|_{L_\infty(T)} \\ &\le \|u - P_h u\|_{L_\infty(T)} + C\epsilon^2 \|b \nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_H)\|_{L_\infty(T)} \\ &\le \|u - P_h u\|_{L_\infty(T)} + C\epsilon^2 \|\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_H)\|_{L_\infty(T)} \\ &\le \|u - P_h u\|_{L_\infty(T)} + C\epsilon^2 \|\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_\infty(T)} \\ &\quad + \epsilon^2 \|\nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L_\infty(T)} \\ &\le \|u - P_h u\|_{L_\infty(T)} + C\epsilon^2 \|\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_\infty(T)} + C\frac{\epsilon^2}{h^2} \|\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(T)}.\end{aligned} \tag{6.15}$$

Using the triangle inequality and Corollary 6.3, we have

$$\begin{aligned}&\|\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(T)} = \|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L_2(T)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(T)} \\ &\le C\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L_2(T_{sd})} + C^s \left(\frac{h}{d}\right)^s \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(T_{sd})} + C\epsilon \|\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_2(T_{sd})}.\end{aligned}$$

Plugging the above inequality into (6.15), we obtain

$$\begin{aligned}\|u - u_h\|_{L_\infty(T)} &\le C(\|u - P_h u\|_{L_\infty(T)} + \epsilon^2 \|\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_\infty(T)}) \\ &\quad + C\left(\frac{\epsilon}{h}\right)^2 \Big(\|\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}\|_{L_2(T_{sd})} \\ &\quad + C^s \left(\frac{h}{d}\right)^s \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(T_{sd})} + \epsilon \|\nabla \cdot (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{L_2(T_{sd})} \Big).\end{aligned}$$

This completes the proof for (6.14). ∎

Different choices of d and s and the local regularities of the solution will lead to different estimates. For example, the following corollary gives an error estimate when $d = \sqrt{h}$ and $s = 4$ for the lowest approximation spaces ($k = 0$).

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COROLLARY 6.6. *Assume that* $|\boldsymbol{\sigma}|_{W^1_\infty(T_{4\sqrt{h}})} \sim \mathcal{O}(1)$. *For* $d = \sqrt{h}$, $s = 4$, *and* $k = 0$, *we have*

$$\begin{aligned}\|u - u_h\|_{L_\infty(T)} &\le C\Big(h|u|_{W^1_\infty(T)} + \epsilon^2\|\nabla\cdot\boldsymbol{\sigma}\|_{L_\infty(T)} + \frac{\epsilon^2}{h}|\boldsymbol{\sigma}|_{W^1_2(T_{4\sqrt{h}})}\Big)\\ &\quad + C^4\epsilon^2\Big(\|\boldsymbol{\sigma}\|_{L_2(\Omega)} + \epsilon\|\sqrt{b}\nabla\cdot\boldsymbol{\sigma}\|_{L_2(\Omega)}\Big) + C\frac{\epsilon^3}{h^2}\|\nabla\cdot\boldsymbol{\sigma}\|_{L_2(T_{4\sqrt{h}})}\\ &\le C^4 h. \end{aligned} \tag{6.16}$$

Remark 6.3. The local maximum error estimate (6.16) shows that the pollution effect is weak in the singularly perturbed problem. If the solution is well behaving near the region of interest, i.e., $|\boldsymbol{\sigma}|_{W^1_\infty(T_{4\sqrt{h}})} \sim \mathcal{O}(1)$, then the approximate solution is guaranteed to achieve the optimal rate of convergence for $\epsilon < h$. The pollution effect is decreasing as ϵ is getting smaller.

Proof. Note that $(\frac{h}{d})^s = h^2$ for $s = 4$ and $d = \sqrt{h}$. Using the approximation properties (2.10), (2.12), the commuting diagram property (2.8) with the stability (2.11), and Theorem 4.1 in (6.14), we have

$$\begin{aligned}\|u - u_h\|_{L_\infty(T)} &\le Ch|u|_{W^1_\infty(T)} + \epsilon^2\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|_{L_\infty(T)} + C^4\frac{\epsilon^2}{h}|\boldsymbol{\sigma}|_{W^1_2(T_{4\sqrt{h}})}\\ &\quad + C\epsilon^2(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(T_{4\sqrt{h}})}) + C\frac{\epsilon^3}{h^2}\|\nabla\cdot(\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma})\|_{L_2(T_{4\sqrt{h}})}\\ &\le Ch|u|_{W^1_\infty(T)} + \epsilon^2\|\nabla\cdot\boldsymbol{\sigma}\|_{L_\infty(T)} + C\frac{\epsilon^2}{h}|\boldsymbol{\sigma}|_{W^1_2(T_{4\sqrt{h}})}\\ &\quad + C^4\epsilon^2(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(T_{4\sqrt{h}})}) + C\frac{\epsilon^3}{h^2}\|\nabla\cdot\boldsymbol{\sigma}\|_{L_2(T_{4\sqrt{h}})}\\ &\le Ch|u|_{W^1_\infty(T)} + \epsilon^2\|\nabla\cdot\boldsymbol{\sigma}\|_{L_\infty(T)} + C\frac{\epsilon^2}{h}|\boldsymbol{\sigma}|_{W^1_2(T_{4\sqrt{h}})}\\ &\quad + C^4\epsilon^2(\|\boldsymbol{\sigma}\|_{L_2(\Omega)} + \epsilon\|\sqrt{b}\nabla\cdot\boldsymbol{\sigma}\|_{L_2(\Omega)}) + C\frac{\epsilon^3}{h^2}\|\nabla\cdot\boldsymbol{\sigma}\|_{L_2(T_{4\sqrt{h}})}.\end{aligned}$$

Now the last inequality in (6.16) follows from the assumption $|\boldsymbol{\sigma}|_{W^1_2(T_{4\sqrt{h}})} \sim \mathcal{O}(1)$ and $\|\boldsymbol{\sigma}\|_{L_2(\Omega)} \sim \mathcal{O}(\epsilon^{-1/2})$ and $\|\nabla\cdot\boldsymbol{\sigma}\|_{L_2(\Omega)} \sim \mathcal{O}(\epsilon^{-3/2})$; see [27]. This completes the proof. ∎

7. Numerical examples. In this section, we present numerical examples confirming our theoretical results. We consider the following model problem:

$$-\epsilon^2\triangle u + u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

We choose the true solution $u = \tanh(\frac{1}{\epsilon}(x^2 + y^2 - \frac{1}{4})) - \tanh(\frac{3}{4\epsilon})$ on the unit circle $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. For our numerical experiments, we consider the problems with $\epsilon = 0.0001$ and $\epsilon = 0.000001$. Note that there is an interior layer at $r = \sqrt{x^2 + y^2} = 1/2$. We use the 16-point quadrature rule [40] for our computations.

We compare our methods with the standard Galerkin methods. For the approximate spaces for our proposed method, we use the lowest Raviart–Thomas spaces RT_0 for the dual variables and piecewise constant spaces V_h^0 for the primary function. The

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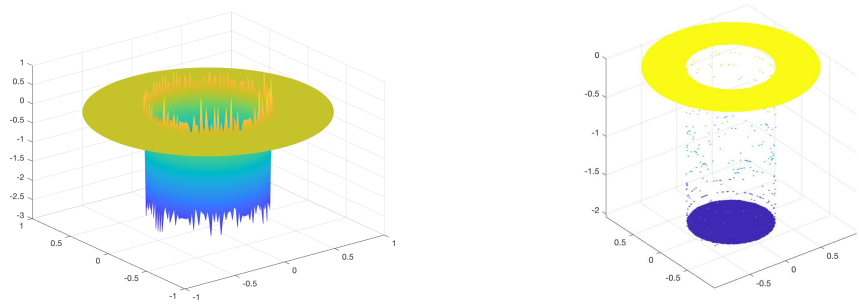

FIG. 7.1. *Left: approximate solution of the standard Galerkin methods, Right: approximate solution of the proposed methods with* $\epsilon = 0.0001$ *and* $h = \frac{1}{2^6}$.

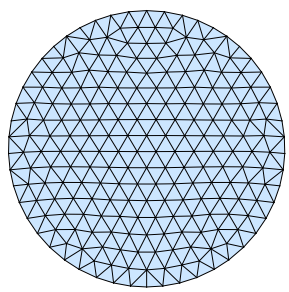

FIG. 7.2. *Mesh with* $h = \frac{1}{8}$.

standard continuous piecewise linear functions are used for the Galerkin method. Both of the approximations shown in Figure 7.1 are obtained on the same mesh with mesh size $h = \frac{1}{64}$. In Figure 7.2, we show the mesh for mesh size $h = \frac{1}{8}$. The mesh is generated by the MATLAB function described in [31]. For the approximate solution for the primary variable u, our solution does not show significant numerical oscillations, while the standard Galerkin method shows solutions with oscillations. To observe the numerical oscillation of the approximate solution, the maximum and minimum values of the approximate solutions are provided in Table 7.1. The maximum and minimum of the true solution u are 0 and -2. While our new method produces numerical solutions without numerical oscillation, the standard Galerkin methods produce solutions with oscillations.

We provide convergence behavior of our approximation u_h and the (local) L_2 projection $P_h u$ in Table 7.2. The accuracy of our approximate solution is comparable to the accuracy of the L_2 projection. Also, we report the convergence behaviors of $\|u - u_h\|_{L_\infty(D)}$, where $D = B(0, \frac{1}{2} - 2 * h), B(0, \frac{3}{8})$, and $B(0, \frac{1}{4})$, where

$$B(0, r) = \{(x, y);\ |x^2 + y^2| \leq r^2\}.$$

Corollary 6.6 predicts that the local region needs to be away from the interior layer, to overcome the pollution effects, i.e., $|\boldsymbol{\sigma}|_{W^1_\infty(T_{4\sqrt{h}})} \sim \mathcal{O}(1)$. The distance is a relative distance proposal to $\sqrt{h}$. Note that $B(0, \frac{1}{2} - 2 * h)$ is close to the (interior) layer at $r = \frac{1}{2}$ in the relative distance $\sqrt{h}$. As a result, the accuracy of the approximate solution is deteriorating as h is getting smaller. On the other hand, the region $B(0, \frac{1}{4})$

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TABLE 7.1
Maximum and minimum of the approximate solutions for $h = \frac{1}{2^6}$.

	u	u_h^P		u_h^G	
		$\epsilon = 0.0001$	$\epsilon = 0.000001$	$\epsilon = 0.0001$	$\epsilon = 0.000001$
Max	0	3.897e-03	1.169e-08	0.9391	0.8932
Min	-2	-2.00 - 0.0389	-2.00 - 1.161e-08	-2.00 - 0.7839	-2.00 - 0.7834

TABLE 7.2
Convergence behaviors with $\epsilon = 0.0001$.

$\frac{1}{h}$	2^3	2^4	2^5	2^6
$\|u - u_h\|_{L_2(\Omega)}$	0.45586	0.30936	0.21449	0.14683
$\|u - P_h u\|_{L_2(\Omega)}$	0.45583	0.30935	0.21443	0.14668
$\|u - u_h\|_{L_\infty(B(0,\frac{1}{2}-2*h))}$	3.8170e-08	2.3114e-07	7.9453e-07	2.8551e-06
$\|u - u_h\|_{L_\infty(B(0,\frac{3}{8}))}$	2.3886e-06	2.3114e-07	1.2808e-10	5.7731e-15
$\|u - u_h\|_{L_\infty(B(0,\frac{1}{4}))}$	3.8170e-08	3.7887e-11	4.8850e-15	4.4409e-15

can be considered as far away from the interior layer, and we observe the improved accuracy as h becomes smaller. The convergence on $B(0, \frac{3}{8})$ shows a somewhat mixed performance. It shows a slight improvement in accuracy as $h = \frac{1}{2^3} \to \frac{1}{2^4}$. This is due to the fact that the region is not far away from the interior layer with respect to the relative distance $\sqrt{h}$ for $h = \frac{1}{2^4}$. After that, it shows similar improvements as in the case for $B(0, \frac{1}{4})$.

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