CALCOLO 36, 215 – 232 (1999)

CALCOLO

A stable nonconforming quadrilateral finite element method for the stationary Stokes and Navier–Stokes equations

Zhiqiang Cai[1,⋆], **Jim Douglas, Jr.**[1,⋆], **Xiu Ye**[2]

[1] Department of Mathematics, Purdue University, 1395 Mathematical Sciences Building, West Lafayette, IN 47907-1395, USA
e-mail: zcai@math.purdue.edu

[2] Department of Mathematics and Statistics, University of Arkansas at Little Rock, Little Rock, AR 72204, USA

Received: January 1999 / Accepted: April 1999

Abstract. Recently, Douglas et al. [4] introduced a new, low-order, nonconforming rectangular element for scalar elliptic equations. Here, we apply this element in the approximation of each component of the velocity in the stationary Stokes and Navier–Stokes equations, along with a piecewise-constant element for the pressure. We obtain a stable element in both cases for which optimal error estimates for the approximation of both the velocity and pressure in L^2 can be established, as well as one in a broken H^1-norm for the velocity.

1 Introduction

In [3], Crouzeix and Raviart considered nonconforming finite element approximations for solving the stationary incompressible Stokes equations. Their low-order, nonconforming simplicial elements consist of standard nonconforming P_1 simplicial elements for the velocity and piecewise constants for the pressure. They showed that this combination is stable and yields first order accuracy. A comparison with the existing first-order conforming simplicial elements (see [1], [5] and references therein) shows that the degrees of freedom and nonzero entries of the coefficient matrix for the nonconforming method are significantly fewer than those for conforming methods. It is natural to consider an analogue for rectangular elements.

⋆ This work was sponsored in part by the NSF and the ONR.

We will adopt a nonconforming rectangular element proposed recently by Douglas et al. [4] for the velocity, and piecewise constants, as in [3], for the pressure. We prove that this choice is stable and gives first order accuracy for both the Stokes and the Navier–Stokes equations. For the Stokes equations, Rannacher and Turek [9] have shown that the “rotated” bilinear basis can be used for the velocity in combination with a piecewise-constant basis for the pressure, also with optimal order approximation for rectangular elements; as was seen in [4], the element we shall use behaves somewhat better when quadrilateral elements are used in the partition of the domain than the rotated bilinears. It is also the case that the analysis is much simpler for our element, and there is no difference in coding or computational efforts associated with replacing the rotated bilinears with our element.

Han [6] proposed an element similar to ours, but with an extra degree of freedom for each component of the velocity, and obtained stability and an analogue of part of our convergence results. The computational procedure associated with our element is slightly simpler and more efficient than that for his element.

In practice, it is possible to partition the domain using both rectangular (wherever possible) and simplicial elements. An inspection of the proofs will show that the corresponding nonconforming finite element approximation retains the same accuracy as mentioned above.

Let Ω be a bounded, open subset in R^d ($d = 2$ or 3) with Lipschitz boundary $\partial\Omega$. We consider the stationary Stokes ($\gamma = 0$) and Navier–Stokes ($\gamma = 1$) equations in dimensionless variables:

$$-\nu\Delta\mathbf{u} + \gamma\sum_{j=1}^{d} u_j\partial_j\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1.1a}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{1.1b}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{1.1c}$$

where the symbols Δ, ∇, and $\nabla\cdot$ denote the Laplacian, gradient, and divergence operators, respectively; $\partial_j = \frac{\partial}{\partial x_j}$; and $\mathbf{f}(x)$ is the unit external volumetric force acting on the fluid at $x \in \Omega$.

We will employ standard definitions for the Sobolev spaces $H^s(\Omega)^d$ and their associated inner products $(\cdot,\cdot)_{s,\Omega}$, norms $\|\cdot\|_{s,\Omega}$, and seminorms $|\cdot|_{s,\Omega}$, $s \geq 0$. (We suppress the designation d on the inner products and norms because dependence on dimension will be clear by context.) The space $H^0(\Omega)^d$ coincides with $L^2(\Omega)^d$, in which case the norm and inner product are denoted by $\|\cdot\|_\Omega$ and $(\cdot,\cdot)_\Omega$, respectively. Finally, let $L_0^2(\Omega)$ denote the subspace of $L^2(\Omega)$ consisting of the functions in $L^2(\Omega)$ having mean value zero. Then, the variational formulation of (1.1) is to find a pair

$(\mathbf{u}, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ such that (cf. [5])

$$a(\mathbf{u}, \mathbf{v}) + \gamma a_s(\mathbf{u}; \mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), \quad \forall\, \mathbf{v} \in H_0^1(\Omega)^d, \tag{1.2a}$$

$$b(\mathbf{u}, q) = 0, \qquad \forall\, q \in L_0^2(\Omega), \tag{1.2b}$$

where the bilinear forms are defined by

$$a(\mathbf{u}, \mathbf{v}) = \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) = \nu \sum_{j=1}^{d} (\nabla u_j, \nabla v_j) \quad \text{and} \qquad b(\mathbf{v}, q) = (\nabla \cdot \mathbf{v}, q),$$

and the trilinear form by

$$a_s(\mathbf{u}; \mathbf{v}, \mathbf{w}) = \frac{1}{2}[a_1(\mathbf{u}; \mathbf{v}, \mathbf{w}) - a_1(\mathbf{u}; \mathbf{w}, \mathbf{v})]$$

$$\text{with} \quad a_1(\mathbf{u}; \mathbf{v}, \mathbf{w}) = \left(\sum_{j=1}^{d} u_j \partial_j \mathbf{v}, \mathbf{w} \right).$$

For fixed $\mathbf{u}$, note that $a_s(\mathbf{u}; \mathbf{v}, \mathbf{w})$ is the skew-symmetric part of $a_1(\mathbf{u}; \mathbf{v}, \mathbf{w})$.

Let $\mathcal{D} = \{\mathbf{v} \in H_0^1(\Omega)^d \,:\, \nabla \cdot \mathbf{v} = 0\}$ denote the divergence-free subspace of $H_0^1(\Omega)^d$. Then, the solution $\mathbf{u}$ of (1.2) lies in $\mathcal{D}$ and satisfies

$$a(\mathbf{u}, \mathbf{v}) + \gamma a_s(\mathbf{u}; \mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \qquad \mathbf{v} \in \mathcal{D}. \tag{1.3}$$

This note is organized as follows. Nonconforming rectangular elements in two dimensions are described in Sect. 2. An *inf-sup* condition for the discrete analogue of the bilinear form $b(\cdot\,,\,\cdot)$ is demonstrated in Sect. 3. Optimal order error estimates are obtained in Sect. 4 with respect to a broken H^1-norm for the velocity and the L^2-norm for the pressure; in addition, an optimal order estimate in the L^2-norm is established for the velocity in Sect. 5. In Sect. 6, we show that all results demonstrated for rectangular partitions in two dimensions extend to partitions into quadrilaterals. Finally, we discuss three-dimensional nonconforming rectangular elements in Sect. 7.

2 Two-dimensional nonconforming rectangular elements

In the next five sections, we restrict to two dimensions and apply the nonconforming element introduced in [4]. The extension of the method to three dimensions is straightforward and will be discussed briefly in Sect. 7.

Let the reference element be the square $\hat{K} = [-1, 1] \times [-1, 1]$. The rotated Q_1 nonconforming element built on $\mathcal{R} =$ Span $\{1, \hat{x}_1, \hat{x}_2, \hat{x}_1^2 - \hat{x}_2^2\}$ does not satisfy the orthogonality relation (6.1) of [4] that plays a critical role in the error analysis there. This failure was remedied in [4] by modifying

$\hat{x}_1^2 - \hat{x}_2^2$ to $(\hat{x}_1^2 - \frac{5}{3}\hat{x}_1^4) - (\hat{x}_2^2 - \frac{5}{3}\hat{x}_2^4)$. Hence, our nonconforming rectangular element for the velocity will be based on

$$\mathcal{Q}(\hat{K}) = \text{ Span } \left\{ 1, \hat{x}_1, \hat{x}_2, \left(\hat{x}_1^2 - \frac{5}{3}\hat{x}_1^4 \right) - \left(\hat{x}_2^2 - \frac{5}{3}\hat{x}_2^4 \right) \right\}.$$

Denote the middle points of edges and the associated edges of the reference element $\hat{K}$ by $\hat{a}_1 = (1, 0)$, $\hat{a}_2 = (0, 1)$, $\hat{a}_3 = (-1, 0)$, and $\hat{a}_4 = (0, -1)$ and $\hat{e}_1, \hat{e}_2, \hat{e}_3$, and $\hat{e}_4$, respectively. Then, the corresponding nodal basis functions associated with the nodes $\hat{a}_i$ have the forms

$$\begin{aligned}
\hat{\phi}_1(\hat{x}) &= \frac{1}{4} + \frac{1}{2}\hat{x}_1 - \frac{3}{8}\left(\left(\hat{x}_1^2 - \frac{5}{3}\hat{x}_1^4 \right) - \left(\hat{x}_2^2 - \frac{5}{3}\hat{x}_2^4 \right) \right), \\
\hat{\phi}_2(\hat{x}) &= \frac{1}{4} + \frac{1}{2}\hat{x}_2 + \frac{3}{8}\left(\left(\hat{x}_1^2 - \frac{5}{3}\hat{x}_1^4 \right) - \left(\hat{x}_2^2 - \frac{5}{3}\hat{x}_2^4 \right) \right), \\
\hat{\phi}_3(\hat{x}) &= \frac{1}{4} - \frac{1}{2}\hat{x}_1 - \frac{3}{8}\left(\left(\hat{x}_1^2 - \frac{5}{3}\hat{x}_1^4 \right) - \left(\hat{x}_2^2 - \frac{5}{3}\hat{x}_2^4 \right) \right), \\
\hat{\phi}_4(\hat{x}) &= \frac{1}{4} - \frac{1}{2}\hat{x}_2 + \frac{3}{8}\left(\left(\hat{x}_1^2 - \frac{5}{3}\hat{x}_1^4 \right) - \left(\hat{x}_2^2 - \frac{5}{3}\hat{x}_2^4 \right) \right).
\end{aligned}$$

It is easy to check that, for $i, j = 1, 2, 3, 4$,

$$\int_{\hat{e}_i} \hat{\phi}_j \, d\hat{s} = \delta_{ij} \left| \hat{e}_i \right|, \tag{2.1}$$

where δ_{ij} is the Kronecker symbol and $\left| \hat{e}_i \right|$ is the length of the edge $\hat{e}_i$.

Let $\bar{\Omega} = \cup_{j=1}^{J} \bar{\Omega}_j$ be a quasiregular rectangular partition of Ω with $\text{diam}(\Omega_j) \leq h$. Denote the boundary edge of Ω_j by $\Gamma_j = \partial\Omega \cap \partial\Omega_j$, the interface between elements Ω_j and Ω_k by

$$\Gamma_{jk} = \Gamma_{kj} = \partial\Omega_j \cap \partial\Omega_k,$$

and the centers of Γ_j and Γ_{jk} by ξ_j and ξ_{jk}, respectively. For each Ω_j, denote by

$$F_j : \hat{x} \to F_j(\hat{x}) = B_j\hat{x} + b_j, \quad B_j \in \mathcal{L}(R^2), \quad b_j \in R^2,$$

the affine, invertible mapping such that $F_j(\hat{K}) = \Omega_j$; the matrix B_j can be assumed diagonal for rectangular Ω_j. Let

$$\mathcal{Q}(\Omega_j) = \{ v : v = \hat{v} \circ F_j^{-1}, \ \hat{v} \in \mathcal{Q}(\hat{K}) \}.$$

The nonconforming rectangular finite element space $\mathcal{NC}^h$ for the velocity will be taken to be

$$\mathcal{NC}^h = \big\{ \mathbf{v} : \mathbf{v}_j = \mathbf{v}|_{\Omega_j} \in \mathcal{Q}(\Omega_j) \times \mathcal{Q}(\Omega_j), \\ \mathbf{v}_j(\xi_{jk}) = \mathbf{v}_k(\xi_{kj}), \mathbf{v}(\xi_j) = 0,\ \forall\, j, k \big\}.$$

Let $\mathcal{P}_0(E)$ denote the space of constants on the set E. The pressure will be approximated by the piecewise-constant functions

$$P^h = \left\{ q \in L_0^2(\Omega) : q|_{\Omega_j} \in \mathcal{P}_0(\Omega_j),\ \forall j \right\}.$$

For any $\mathbf{v} = (v_1, v_2)^t$ in $\mathcal{NC}^h$, it is easy to verify that

$$\int_{\Gamma_{jk}} [v_i]\, ds = 0 \quad \text{and} \quad \int_{\Gamma_j} v_i\, ds = 0, \quad i = 1, 2, \tag{2.2}$$

where $[v_i] = v_i|_{\Gamma_{jk}} - v_i|_{\Gamma_{kj}}$ denotes the jump of the function v_i across Γ_{jk}; (2.2) expresses the orthogonalities that were important in [4].

Let $(\cdot,\cdot)_j = (\cdot,\cdot)_{\Omega_j}$ and $\langle f, g\rangle_j = \int_{\partial\Omega_j} fg\, ds$, and define the discrete counterparts of the bilinear and trilinear forms as follows:

$$a_h(\mathbf{u}, \mathbf{v}) = \nu \sum_j (\nabla\mathbf{u}, \nabla\mathbf{v})_j, \quad a_{1,h}(\mathbf{u}; \mathbf{v}, \mathbf{w}) = \sum_j \left(\sum_{i=1}^{2} u_i \partial_i \mathbf{v}, \mathbf{w} \right)_j,$$

$$b_h(\mathbf{v}, q) = \sum_j (\nabla \cdot \mathbf{v}, q)_j,$$

and

$$a_{s,h}(\mathbf{u}; \mathbf{v}, \mathbf{w}) = \frac{1}{2} \left[a_{1,h}(\mathbf{u}; \mathbf{v}, \mathbf{w}) - a_{1,h}(\mathbf{u}; \mathbf{w}, \mathbf{v}) \right].$$

For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1(\Omega_j)^d$ with $1 \le j \le J$, integration by parts on each element gives

$$a_{1,h}(\mathbf{u}; \mathbf{v}, \mathbf{w}) = -a_{1,h}(\mathbf{u}; \mathbf{w}, \mathbf{v}) - \sum_j ((\nabla \cdot \mathbf{u})\mathbf{v}, \mathbf{w})_j + \sum_j \langle (\mathbf{u} \cdot \mathbf{n}_j)\mathbf{v}, \mathbf{w} \rangle_j$$

where $\mathbf{n}_j$ is the unit outward normal to $\partial\Omega_j$. Hence,

$$a_{1,h}(\mathbf{u}; \mathbf{v}, \mathbf{w}) = a_{s,h}(\mathbf{u}; \mathbf{v}, \mathbf{w}) - \frac{1}{2} \sum_j ((\nabla \cdot \mathbf{u})\mathbf{v}, \mathbf{w})_j + \frac{1}{2} \sum_j \big\langle (\mathbf{u} \cdot \mathbf{n}_j)\mathbf{v}, \mathbf{w} \big\rangle_j. \tag{2.3}$$

It is known (cf. [5]) that the trilinear forms $a_1(\mathbf{u}; \mathbf{v}, \mathbf{w})$ and $a_s(\mathbf{u}; \mathbf{v}, \mathbf{w})$ are continuous in $(H^1(\Omega)^2)^3$. The same argument applied on each Ω_j implies that $a_{1,h}(\mathbf{u}; \mathbf{v}, \mathbf{w})$ and $a_{s,h}(\mathbf{u}; \mathbf{v}, \mathbf{w})$ are also continuous, i.e.,

$$a_{1,h}(\mathbf{u}; \mathbf{v}, \mathbf{w}),\ a_{s,h}(\mathbf{u}; \mathbf{v}, \mathbf{w}) \le C \|\mathbf{u}\|_{1,h} \|\mathbf{v}\|_{1,h} \|\mathbf{w}\|_{1,h}, \tag{2.4}$$

for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1(\Omega_j)^2$ with $1 \le j \le J$. Here, $\|\cdot\|_{1,h}$ denotes the (broken) energy semi-norm

$$\|\mathbf{v}\|_{1,h} = \sqrt{a_h(\mathbf{v}, \mathbf{v})}\,.$$

By (2.2), $\|\cdot\|_{1,h}$ is a norm over $\mathcal{NC}^h$.

The nonconforming finite element approximation of (1.2) is to find a pair $(\mathbf{u}_h, p_h) \in \mathcal{NC}^h \times P^h$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}) + \gamma a_{s,h}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}) - b_h(\mathbf{v}, p_h) = (\mathbf{f}, \mathbf{v}), \quad \mathbf{v} \in \mathcal{NC}^h, \tag{2.5a}$$

$$b_h(\mathbf{u}_h, q) = 0, \qquad q \in P^h. \tag{2.5b}$$

Let $\mathcal{D}^h$ denote the discrete divergence-free subspace of $\mathcal{NC}^h$, i.e.,

$$\mathcal{D}^h = \{\mathbf{v} \in \mathcal{NC}^h \,:\, b_h(\mathbf{v}, q) = 0, \quad \forall q \in P^h\}.$$

Then the solution $\mathbf{u}_h$ of the above problem lies in $\mathcal{D}^h$ and satisfies

$$a_h(\mathbf{u}_h, \mathbf{v}) + \gamma a_{s,h}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall\, \mathbf{v} \in \mathcal{D}^h. \tag{2.6}$$

3 The inf-sup condition

It is well-known (see, e.g., [5]) that the bilinear form $b(\cdot,\cdot)$ satisfies the *inf-sup* condition, i.e., there exists a positive constant ρ such that

$$\sup_{\mathbf{v} \in H_0^1(\Omega)^d} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_1} \ge \rho \|q\|, \quad \forall\, q \in L_0^2(\Omega). \tag{3.1}$$

We follow the argument in Crouzeix and Raviart [3] to show that the bilinear form $b_h(\cdot,\cdot)$ satisfies a discrete *inf-sup* condition on $\mathcal{NC}^h \times P^h$, i.e., there exists a positive constant β, independent of the mesh size h, such that

$$\sup_{\mathbf{v} \in \mathcal{NC}^h} \frac{b_h(\mathbf{v}, q)}{\|\mathbf{v}\|_{1,h}} \ge \beta \|q\|, \quad \forall\, q \in P^h. \tag{3.2}$$

Denote the edges of Ω_j by e_j^i for $i = 1, 2, 3, 4$, and the midpoint of the edge e_j^i by a_j^i. Define the operator $\pi_j : H^1(\Omega_j) \to \mathcal{Q}(\Omega_j)$ by requiring that, for any $v \in H^1(\Omega_j)$,

$$\int_{e_j^i} \pi_j v \, ds = \int_{e_j^i} v \, ds, \quad \text{for} \quad i = 1, 2, 3, 4. \tag{3.3}$$

Since (2.1) is invariant under the mapping F_j^{-1}, (3.3) determines the midpoint values of $\pi_j v$ as

$$\pi_j v(a_j^i) = \frac{1}{|e_j^i|} \int_{e_j^i} v \, ds, \qquad i = 1, 2, 3, 4.$$

Therefore, π_j reproduces $\mathcal{Q}(\Omega_j)$. By a standard Bramble-Hilbert argument,

$$|\pi_j v - v|_{1,j} \le Ch^m |v|_{m+1,j}, \quad \forall\, v \in H^{m+1}(\Omega_j), \quad m = 0, 1; \tag{3.4}$$

thus,

$$\|\pi_j v\|_{1,j} \le C\|v\|_{1,j}. \tag{3.5}$$

(We use C with or without subscripts in this note to denote a generic positive constant, possibly different at different occurrences, that is independent of the mesh size h but may depend on the domain Ω.)

For any $\mathbf{v} \in H_0^1(\Omega)^2$, define $\Pi_h \mathbf{v} \in \mathcal{NC}^h$ by

$$(\Pi_h \mathbf{v})_i|_{\Omega_j} = \pi_j v_i, \quad \forall\, j, \quad i = 1, 2. \tag{3.6}$$

Lemma 3.1 *The operator* $\Pi_h : H_0^1(\Omega)^2 \to \mathcal{NC}^h$ *has the following properties:*

$$b_h(\Pi_h \mathbf{v} - \mathbf{v}, q) = 0, \quad q \in P^h, \tag{3.7}$$

$$\|\Pi_h \mathbf{v}\|_{1,h} \le C\|\mathbf{v}\|_1, \quad \mathbf{v} \in H_0^1(\Omega)^2. \tag{3.8}$$

Proof Let $\mathbf{n}_j = (n_{1,j}, n_{2,j})^t$ be the outward unit normal on $\partial\Omega_j$ and set $q_j = q|_{\Omega_j}$ for any $q \in P^h$. By the divergence theorem,

$$\begin{aligned} b_h(\Pi_h \mathbf{v} - \mathbf{v}, q) &= \sum_j q_j \int_{\Omega_j} \operatorname{div}(\Pi_h \mathbf{v} - \mathbf{v})\, dx \\ &= \sum_j q_j \int_{\partial\Omega_j} (\Pi_j \mathbf{v} - \mathbf{v}) \cdot \mathbf{n}_j\, ds \\ &= \sum_j q_j \int_{\partial\Omega_j} [(\pi_j v_1 - v_1) n_{1,j} + (\pi_j v_2 - v_2) n_{2,j}]\, ds, \end{aligned}$$

so that (3.7) follows from the definition of π_j. Also, (3.8) is a straightforward consequence of the definition of Π_h and (3.5). □

We can now establish (3.2). For any $q \in P^h \subset L_0^2(\Omega)$, it follows from Lemma 3.1 that

$$\begin{aligned} \sup_{\mathbf{v} \in \mathcal{NC}^h} \frac{b_h(\mathbf{v}, q)}{\|\mathbf{v}\|_{1,h}} &\ge \sup_{\mathbf{w} \in H_0^1(\Omega)^2} \frac{b_h(\Pi_h \mathbf{w}, q)}{\|\Pi_h \mathbf{w}\|_{1,h}} = \sup_{\mathbf{w} \in H_0^1(\Omega)^2} \frac{b(\mathbf{w}, q)}{\|\Pi_h \mathbf{w}\|_{1,h}} \\ &\ge C \sup_{\mathbf{w} \in H_0^1(\Omega)^2} \frac{b(\mathbf{w}, q)}{\|\mathbf{w}\|_1}. \end{aligned}$$

Combining this relation with the *inf-sup* condition (3.1) implies (3.2).

Proposition 3.1 *The bilinear form* $b_h(\cdot, \cdot)$ *satisfies the discrete inf-sup condition* (3.2) *in* $\mathcal{NC}^h \times P^h$, *where* β *is a positive constant independent of the mesh size* h.

4 Error estimates derived from stability

Optimal order error estimates in the (broken) energy norm for the velocity and the L^2-norm for the pressure will be derived, with the analysis of the error in the velocity being based on (1.3) and (2.6). Then, the discrete *inf-sup* condition (3.2) will be used to estimate the error of the pressure approximation. Later, in Sect. 6, a standard duality argument gives an error estimate for the velocity in L^2.

Let

$$\Lambda^h = \big\{\lambda \,:\, \lambda_{jk} = tr_{\Gamma_{jk}}(\lambda|_{\Omega_j}) \in \mathcal{P}_0(\Gamma_{jk}); \\ \lambda_{jk} + \lambda_{kj} = 0; \lambda_j = tr_{\Gamma_j}(\lambda|_{\Omega_j}) \in \mathcal{P}_0(\Gamma_j)\big\}.$$

Define projections $R_h : H^2(\Omega)^2 \to \mathcal{NC}^h$ and $P_0 : H^1(\Omega)^2 \to \Lambda^h \times \Lambda^h$ as follows:

$$R_h\mathbf{v}(\xi) = \mathbf{v}(\xi), \qquad \forall\, \xi = \xi_{jk} \text{ or } \xi_j; \tag{4.1}$$

$$\big\langle P_0\mathbf{w}_j, \mathbf{z}\big\rangle_\Gamma = \left\langle \frac{\partial \mathbf{w}_j}{\partial \nu_j}, \mathbf{z}\right\rangle_\Gamma, \quad \forall\, \mathbf{z} \in \mathcal{P}_0(\Gamma)^2, \ \ \forall\, \Gamma = \Gamma_{jk} \text{ or } \Gamma_j, \tag{4.2}$$

for $\mathbf{v} \in H^2(\Omega)^2$ and $\mathbf{w} \in H^1(\Omega)^2$, respectively. Also, define projections $Q_0 : H^1(\Omega) \to \mathcal{P}_0(\Gamma)$ and $S_h : H^1(\Omega) \to P^h$ by

$$\langle Q_0 q, z\rangle_\Gamma = \langle q, z\rangle_\Gamma, \quad \forall\, z \in \mathcal{P}_0(\Gamma), \ \ \forall\, \Gamma = \Gamma_{jk} \text{ or } \Gamma_j; \tag{4.3}$$

$$(S_h q, z) = (q, z), \quad \forall\, z \in P^h, \tag{4.4}$$

for $q \in H^1(\Omega)$. It is easy to verify that P_0 satisfies the following orthogonality:

$$\langle P_0\mathbf{v}_j, \mathbf{w}_j\rangle_{\Gamma_{jk}} + \langle P_0\mathbf{v}_k, \mathbf{w}_k\rangle_{\Gamma_{kj}} = \langle P_0\mathbf{v}_j, \mathbf{w}_j - \mathbf{w}_k\rangle_{\Gamma_{jk}} = 0, \quad \forall\, \mathbf{w} \in \mathcal{NC}^h. \tag{4.5}$$

Since R_h and S_h reproduce linear functions on elements and P_0 and Q_0 reproduce constants on faces, the standard polynomial approximation results imply that

$$\|\mathbf{v} - R_h\mathbf{v}\| + h\left(\sum_j \|\mathbf{v} - R_h\mathbf{v}\|_{1,j}^2\right)^{1/2} + h^2\left(\sum_j \|\mathbf{v} - R_h\mathbf{v}\|_{2,j}^2\right)^{1/2} \\ + h^{1/2}\left(\sum_j |||\mathbf{v} - R_h\mathbf{v}|||_j^2\right)^{1/2} \le Ch^2\|\mathbf{v}\|_2, \quad \mathbf{v} \in H^2(\Omega)^2, \tag{4.6a}$$

$$\left(\sum_j |||\frac{\partial \mathbf{w}}{\partial \nu_j} - P_0\mathbf{w}|||_j^2\right)^{1/2} \le Ch^{1/2}||\mathbf{w}||_2, \quad \mathbf{w} \in H^2(\Omega)^2, \tag{4.6b}$$

$$\|S_h q - q\| + h^{\frac{1}{2}} \left(\sum_j |||q - Q_0 q|||_j^2 \right)^{1/2} \leq Ch\|q\|_1, \quad q \in H^1(\Omega), \quad (4.6c)$$

where $|||\cdot|||_j = \left(\sum_k \|\cdot\|^2_{L^2(\Gamma_{jk})}\right)^{\frac{1}{2}}$ denotes the L^2-norm over the boundary of Ω_j; Γ_j replaces Γ_{jk} for a boundary face.

Following Girault and Raviart [5], define the quantities

$$\mathcal{N} = \sup_{\mathbf{u},\mathbf{v},\mathbf{w} \in H_0^1(\Omega)^d} \frac{a_s(\mathbf{u}; \mathbf{v}, \mathbf{w})}{\|\mathbf{u}\|_1 \|\mathbf{v}\|_1 \|\mathbf{w}\|_1} \quad \text{and}$$
$$\mathcal{N}_h = \sup_{\mathbf{u},\mathbf{v},\mathbf{w} \in \mathcal{NC}^h} \frac{a_{s,h}(\mathbf{u}; \mathbf{v}, \mathbf{w})}{\|\mathbf{u}\|_{1,h} \|\mathbf{v}\|_{1,h} \|\mathbf{w}\|_{1,h}}, \quad (4.7)$$

which are norms for the trilinear forms a_s and $a_{s,h}$, respectively. It is well-known (see, e.g., [5]) that (1.2) has a unique solution if

$$\frac{\gamma \mathcal{N}}{\nu^2} ||\mathbf{f}||_{-1} < 1. \quad (4.8)$$

Hence, we will always assume (4.8). We will also assume throughout that, for $h > 0$,

$$\frac{\gamma \mathcal{N}_h}{\nu^2} \|\mathbf{f}\|_* \leq \zeta < 1, \quad \text{where} \quad ||\mathbf{f}||_* = \sup_{\mathbf{v} \in \mathcal{NC}^h} \frac{(\mathbf{f}, \mathbf{v})}{|\mathbf{v}|_{1,h}}. \quad (4.9)$$

These two assumptions will not be repeated in the statements of the various theorems and lemmas below. Note that they pose no constraints for the Stokes problem. Taking $\mathbf{v} = \mathbf{u}$ and $\mathbf{v} = \mathbf{u}_h$ in (1.3) and (2.6), respectively, and using the facts that $a_s(\mathbf{u}; \mathbf{u}, \mathbf{u}) = a_{s,h}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{u}_h) = 0$, the Cauchy–Schwarz inequality implies that

$$\|\mathbf{u}\|_1 \leq \nu^{-1} ||\mathbf{f}||_1 \quad \text{and} \quad ||\mathbf{u}_h||_{1,h} \leq \nu^{-1} ||\mathbf{f}||_*. \quad (4.10)$$

Lemma 4.1 *Let* $(\mathbf{u}, p)$ *and* $(\mathbf{u}_h, p_h)$ *be the solutions of* (1.2) *and* (2.5), *respectively. Then, for* $\nu \geq \nu^* = \sqrt{\gamma \mathcal{N}_h ||\mathbf{f}||_*}$, *there is a positive constant C such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,h} \leq \left(\inf_{\mathbf{v} \in \mathcal{NC}^h} \|\mathbf{u} - \mathbf{v}\|_{1,h} + \sup_{\mathbf{v} \in \mathcal{D}^h} \frac{|a_h(\mathbf{u}, \mathbf{v}) + \gamma a_{s,h}(\mathbf{u}; \mathbf{u}, \mathbf{v}) - (\mathbf{f}, \mathbf{v})|}{\|\mathbf{v}\|_{1,h}} \right), \quad (4.11)$$

$$\|p - p_h\| \le C\Bigg(\inf_{q\in P^h} \|p-q\| + \sup_{\mathbf{v}\in\mathcal{NC}^h} \frac{|a_h(\mathbf{u},\mathbf{v}) + \gamma a_{s,h}(\mathbf{u};\mathbf{u},\mathbf{v}) - b_h(\mathbf{v},p) - (\mathbf{f},\mathbf{v})|}{\|\mathbf{v}\|_{1,h}} + \inf_{\mathbf{v}\in\mathcal{NC}^h} \|\mathbf{u}-\mathbf{v}\|_{1,h} + \sup_{\mathbf{v}\in\mathcal{D}^h} \frac{|a_h(\mathbf{u},\mathbf{v}) + \gamma a_{s,h}(\mathbf{u};\mathbf{u},\mathbf{v}) - (\mathbf{f},\mathbf{v})|}{\|\mathbf{v}\|_{1,h}} \Bigg). \tag{4.12}$$

Proof For $\mathbf{v} \in \mathcal{D}^h$, it follows from (2.6) that

$$\begin{aligned}
\nu||\mathbf{u}_h - \mathbf{v}||_{1,h}^2 &= a_h(\mathbf{u}_h - \mathbf{v}, \mathbf{u}_h - \mathbf{v}) \\
&= a_h(\mathbf{u}_h, \mathbf{u}_h - \mathbf{v}) - a_h(\mathbf{u}, \mathbf{u}_h - \mathbf{v}) + a_h(\mathbf{u} - \mathbf{v}, \mathbf{u}_h - \mathbf{v}) \\
&= (\mathbf{f}, \mathbf{u}_h - \mathbf{v}) - \gamma a_{s,h}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{u}_h - \mathbf{v}) \\
&\quad - a_h(\mathbf{u}, \mathbf{u}_h - \mathbf{v}) + a_h(\mathbf{u} - \mathbf{v}, \mathbf{u}_h - \mathbf{v}) \\
&= [(\mathbf{f}, \mathbf{u}_h - \mathbf{v}) - a_h(\mathbf{u}, \mathbf{u}_h - \mathbf{v}) - \gamma a_{s,h}(\mathbf{u}; \mathbf{u}, \mathbf{u}_h - \mathbf{v})] \\
&\quad + a_h(\mathbf{u} - \mathbf{v}, \mathbf{u}_h - \mathbf{v}) \\
&\quad + \gamma[a_{s,h}(\mathbf{u}; \mathbf{u}, \mathbf{u}_h - \mathbf{v}) - a_{s,h}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{u}_h - \mathbf{v})].
\end{aligned} \tag{4.13}$$

Since $a_{s,h}(\mathbf{u}; \mathbf{u}_h - \mathbf{v}, \mathbf{u}_h - \mathbf{v}) = 0$, (2.4) and (4.10) give

$$\begin{aligned}
&|a_{s,h}(\mathbf{u}; \mathbf{u}, \mathbf{u}_h - \mathbf{v}) - a_{s,h}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{u}_h - \mathbf{v})| \\
&= |a_{s,h}(\mathbf{u}; \mathbf{u} - \mathbf{v}, \mathbf{u}_h - \mathbf{v}) + a_{s,h}(\mathbf{u} - \mathbf{u}_h; \mathbf{u}_h, \mathbf{u}_h - \mathbf{v})| \\
&= |a_{s,h}(\mathbf{u}; \mathbf{u} - \mathbf{v}, \mathbf{u}_h - \mathbf{v}) + a_{s,h}(\mathbf{u} - \mathbf{v}; \mathbf{u}_h, \mathbf{u}_h - \mathbf{v}) \\
&\quad + a_{s,h}(\mathbf{v} - \mathbf{u}_h; \mathbf{u}_h, \mathbf{u}_h - \mathbf{v})| \\
&\le \mathcal{N}_h \left(\|\mathbf{u}\|_1 + \|\mathbf{u}_h\|_{1,h}\right) ||\mathbf{u} - \mathbf{v}||_{1,h}||\mathbf{u}_h - \mathbf{v}||_{1,h} \\
&\quad + \mathcal{N}_h\|\mathbf{u}_h\|_{1,h}||\mathbf{u} - \mathbf{u}_h||_{1,h}||\mathbf{u}_h - \mathbf{v}||_{1,h} \\
&\le C||\mathbf{u} - \mathbf{v}||_{1,h}||\mathbf{u}_h - \mathbf{v}||_{1,h} + \frac{\mathcal{N}_h}{\nu}||\mathbf{f}||_*||\mathbf{u} - \mathbf{u}_h||_{1,h}||\mathbf{u}_h - \mathbf{v}||_{1,h}.
\end{aligned} \tag{4.14}$$

Using (4.14) and dividing both sides of (4.13) by $||\mathbf{u}_h - \mathbf{v}||_{1,h}$ gives

$$\begin{aligned}
\nu||\mathbf{u}_h - \mathbf{v}||_{1,h} \le \sup_{\mathbf{w}\in\mathcal{D}^h} &\frac{|(\mathbf{f}, \mathbf{w}) - a_h(\mathbf{u}, \mathbf{w}) - \gamma a_{s,h}(\mathbf{u}; \mathbf{u}, \mathbf{w})|}{||\mathbf{w}||_{1,h}} \\
&+ C||\mathbf{u} - \mathbf{v}||_{1,h} + \frac{\gamma\mathcal{N}_h}{\nu}||\mathbf{f}||_*||\mathbf{u} - \mathbf{u}_h||_{1,h}.
\end{aligned}$$

Then,

$$\nu\left(1-\frac{\gamma\mathcal{N}_h}{\nu^2}||\mathbf{f}||_*\right)||\mathbf{u}_h-\mathbf{v}||_{1,h}$$
$$\leq \sup_{\mathbf{w}\in\mathcal{D}^h}\frac{|(\mathbf{f},\mathbf{w})-a_h(\mathbf{u},\mathbf{w})-\gamma a_{s,h}(\mathbf{u};\mathbf{u},\mathbf{w})|}{||\mathbf{w}||_{1,h}}+C||\mathbf{u}-\mathbf{v}||_{1,h},$$

which, together with the triangle inequality, gives an appropriate analogue of the second Strang lemma:

$$||\mathbf{u}-\mathbf{u}_h||_{1,h}$$
$$\leq C\left(\inf_{\mathbf{v}\in\mathcal{D}^h}||\mathbf{u}-\mathbf{v}||_{1,h}+\sup_{\mathbf{w}\in\mathcal{D}^h}\frac{|(\mathbf{f},\mathbf{w})-a_h(\mathbf{u},\mathbf{w})-\gamma a_{s,h}(\mathbf{u};\mathbf{u},\mathbf{w})|}{||\mathbf{w}||_{1,h}}\right).$$

The same proof as one constructed by Girault and Raviart [5] shows that

$$\inf_{\mathbf{v}\in\mathcal{D}^h}\|\mathbf{u}-\mathbf{v}\|_{1,h}\leq C\inf_{\mathbf{v}\in\mathcal{NC}^h}\|\mathbf{u}-\mathbf{v}\|_{1,h}.$$

Now, (4.11) follows from the two inequalities above.

For any $(\mathbf{v},q)\in\mathcal{NC}^h\times P^h$, it follows from (2.5) that

$$\begin{aligned}
&b_h(\mathbf{v},q-p_h)\\
&\quad=b_h(\mathbf{v},q-p)+b_h(\mathbf{v},p)-b_h(\mathbf{v},p_h)\\
&\quad=b_h(\mathbf{v},q-p)+b_h(\mathbf{v},p)+(\mathbf{f},\mathbf{v})-a_h(\mathbf{u}_h,\mathbf{v})-\gamma a_{s,h}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v})\\
&\quad=b_h(\mathbf{v},q-p)+[(\mathbf{f},\mathbf{v})-a_h(\mathbf{u},\mathbf{v})-\gamma a_{s,h}(\mathbf{u};\mathbf{u},\mathbf{v})+b_h(\mathbf{v},p)]\\
&\qquad+a_h(\mathbf{u}-\mathbf{u}_h,\mathbf{v})+\gamma[a_{s,h}(\mathbf{u};\mathbf{u},\mathbf{v})-a_{s,h}(\mathbf{u}_h;\mathbf{u}_h,\mathbf{v})]\\
&\quad=b_h(\mathbf{v},q-p)+[(\mathbf{f},\mathbf{v})-a_h(\mathbf{u},\mathbf{v})-\gamma a_{s,h}(\mathbf{u};\mathbf{u},\mathbf{v})+b_h(\mathbf{v},p)]\\
&\qquad+a_h(\mathbf{u}-\mathbf{u}_h,\mathbf{v})+\gamma[a_{s,h}(\mathbf{u};\mathbf{u}-\mathbf{u}_h,\mathbf{v})+a_{s,h}(\mathbf{u}-\mathbf{u}_h;\mathbf{u}_h,\mathbf{v})].
\end{aligned}$$

It then follows from the triangle inequality, (3.2), the above equality, the boundedness of the bilinear forms $b_h(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$, and (4.10) that

$$\begin{aligned}
\|p-p_h\|&\leq\|p-q\|+\|q-p_h\|\\
&\leq\|p-q\|+\frac{1}{\beta}\sup_{\mathbf{v}\in\mathcal{NC}^h}\frac{|b_h(\mathbf{v},q-p_h)|}{\|\mathbf{v}\|_{1,h}}\\
&\leq C\|p-q\|\\
&\quad+\frac{1}{\beta}\sup_{\mathbf{v}\in\mathcal{NC}^h}\frac{|(\mathbf{f},\mathbf{v})-a_h(\mathbf{u},\mathbf{v})-\gamma a_{s,h}(\mathbf{u};\mathbf{u},\mathbf{v})+b_h(\mathbf{v},p)|}{\|\mathbf{v}\|_{1,h}}\\
&\quad+C\|\mathbf{u}-\mathbf{u}_h\|_{1,h},
\end{aligned}$$

which, together with (4.11), implies (4.12). Hence, the lemma has been proved. □

To bound the truncation errors in (4.11) and (4.12), we follow the proof in [4] to estimate sums of some surface integrals over all edges.

Lemma 4.2 *For any* $\boldsymbol{\phi},\ \mathbf{w} \in H_0^1(\Omega)^2 \cup \mathcal{NC}^h$,

$$\left|\sum_j \left\langle \frac{\partial \mathbf{w}}{\partial \mathbf{n}_j}, \boldsymbol{\phi} \right\rangle_j \right| \le Ch\|\mathbf{w}\|_2 \|\boldsymbol{\phi}\|_{1,h}, \quad \forall\ \mathbf{w} \in H_0^1(\Omega)^2 \cap H^2(\Omega)^2, \tag{4.15}$$

$$\left|\sum_j \left\langle (\mathbf{w}\cdot\mathbf{n}_j)\mathbf{v}, \boldsymbol{\phi} \right\rangle_j \right| \le Ch\|\mathbf{w}\|_{1,h} \|\mathbf{v}\|_2 \|\boldsymbol{\phi}\|_{1,h}, \quad \forall\ \mathbf{v} \in H^2(\Omega)^2, \tag{4.16}$$

$$\left|\sum_j \left\langle q, \boldsymbol{\phi}\cdot\mathbf{n}_j \right\rangle_j \right| \le Ch\|q\|_1 \|\boldsymbol{\phi}\|_{1,h}, \quad \forall\ q \in H^1(\Omega). \tag{4.17}$$

Proof For any $\mathbf{w} \in H_0^1(\Omega)^2 \cap H^2(\Omega)^2$, it follows either from the fact that $P_0\mathbf{w} \in \Lambda^h \times \Lambda^h$ if $\boldsymbol{\phi} \in H_0^1(\Omega)^2$, or from the orthogonality (4.5) if $\boldsymbol{\phi} \in \mathcal{NC}^h$, that

$$\sum_j \left\langle P_0\mathbf{w}, \boldsymbol{\phi} \right\rangle_j = 0.$$

Hence, for $\mathbf{m}_j \in \mathcal{P}_0(\Omega_j)^2$ taken as the average of $\boldsymbol{\phi}$ over Ω_j,

$$\sum_j \left\langle \frac{\partial \mathbf{w}}{\partial \mathbf{n}_j}, \boldsymbol{\phi} \right\rangle_j = \sum_j \left\langle \frac{\partial \mathbf{w}}{\partial \mathbf{n}_j} - P_0\mathbf{w}, \boldsymbol{\phi} \right\rangle_j = \sum_j \left\langle \frac{\partial \mathbf{w}}{\partial \mathbf{n}_j} - P_0\mathbf{w}, \boldsymbol{\phi} - \mathbf{m} \right\rangle_j.$$

Now, (4.15) follows from the approximation property (4.6), the Cauchy–Schwarz inequality, and a standard trace theorem that

$$\begin{aligned}\left|\sum_j \left\langle \frac{\partial \mathbf{w}}{\partial \mathbf{n}_j}, \boldsymbol{\phi} \right\rangle_j \right| &\le Ch^{\frac{1}{2}}\|\mathbf{w}\|_2 \left(\sum_j \|\boldsymbol{\phi} - \mathbf{m}\|_j \|\nabla(\boldsymbol{\phi} - \mathbf{m})\|_j\right)^{\frac{1}{2}} \\ &\le Ch\|\mathbf{w}\|_2 \left(\sum_j \|\nabla\boldsymbol{\phi}\|_j^2\right)^{\frac{1}{2}}.\end{aligned}$$

By (2.2), (4.3), and (4.6),

$$\begin{aligned}\left|\sum_j \langle q, \boldsymbol{\phi}\cdot\mathbf{n}_j\rangle_j\right| &= \left|\sum_j \langle q - Q_0 q, (\boldsymbol{\phi} - \mathbf{m})\cdot\mathbf{n}_j\rangle_j\right| \\ &\le Ch^{\frac{1}{2}}\|q\|_1 \left(\sum_j \|\boldsymbol{\phi} - \mathbf{m}\|_j \|\nabla(\boldsymbol{\phi} - \mathbf{m})\|_j\right)^{\frac{1}{2}} \\ &\le Ch\|q\|_1 \|\boldsymbol{\phi}\|_{1,h},\end{aligned}$$

which proves (4.17). We can prove (4.16) similarly. □

Theorem 4.1 *Let* $(\mathbf{u}, p) \in H^2(\Omega)^2 \times H^1(\Omega)$ *and* $(\mathbf{u}_h, p_h) \in \mathcal{NC}^h \times P^h$ *be the solutions of* (1.2) *and* (2.5)*, respectively. Then,*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,h} + \|p - p_h\| \le Ch(\|\mathbf{u}\|_2 + \|p\|_1). \tag{4.18}$$

Proof Multiply (1.1a) by $\mathbf{v}$ in $\mathcal{NC}^h$, integrate by parts on each element, and use (2.3) to see that

$$\begin{aligned}(\mathbf{f}, \mathbf{v}) &= \left(-\nu\Delta\mathbf{u} + \gamma\sum_{i=1}^{2} u_i\partial_i\mathbf{u} + \nabla p, \mathbf{v}\right)\\ &= a_h(\mathbf{u}, \mathbf{v}) + \gamma a_{s,h}(\mathbf{u}; \mathbf{u}, \mathbf{v}) - b_h(\mathbf{v}, p) - \nu\sum_j\left\langle\frac{\partial\mathbf{u}}{\partial\mathbf{n}_j}, \mathbf{v}\right\rangle_j \\ &\quad + \frac{\gamma}{2}\sum_j\langle(\mathbf{u}\cdot\mathbf{n}_j)\mathbf{u}, \mathbf{v}\rangle_j + \sum_j\langle p, \mathbf{v}\cdot\mathbf{n}_j\rangle_j.\end{aligned} \tag{4.19}$$

Rearranging (4.19) gives

$$\begin{aligned}&a_h(\mathbf{u}, \mathbf{v}) + \gamma a_{s,h}(\mathbf{u}; \mathbf{u}, \mathbf{v}) - (\mathbf{f}, \mathbf{v})\\ &\quad = b_h(\mathbf{v}, p) + \nu\sum_j\left\langle\frac{\partial\mathbf{u}}{\partial\mathbf{n}_j}, \mathbf{v}\right\rangle_j - \frac{\gamma}{2}\sum_j\langle(\mathbf{u}\cdot\mathbf{n}_j)\mathbf{u}, \mathbf{v}\rangle_j - \sum_j\langle p, \mathbf{v}\cdot\mathbf{n}_j\rangle_j.\end{aligned}$$

By (4.10), (4.6), the triangle inequality, and Lemma 4.2, it suffices to show that

$$|b_h(\mathbf{v}, p)| \le Ch\|p\|_1\|\mathbf{v}\|_{1,h}, \quad \forall\, \mathbf{v} \in \mathcal{D}^h.$$

This is an immediate consequence of the fact that $b_h(\mathbf{v}, p) = b_h(\mathbf{v}, p - S_h p)$ for all $\mathbf{v} \in \mathcal{D}^h$, the Cauchy–Schwarz inequality, and (4.6). Thus, the theorem is proved. □

5 Duality and the L^2-error estimate

We consider the linear dual problem

$$\begin{cases} -\nu\Delta\boldsymbol{\psi} - \sum_{j=1}^{2} u_j\partial_j\boldsymbol{\psi} + \sum_{j=1}^{2}\psi_j\nabla u_j + \nabla\chi = \mathbf{u} - \mathbf{u}_h & \text{in } \Omega,\\ \nabla\cdot\boldsymbol{\psi} = 0 & \text{in } \Omega,\\ \boldsymbol{\psi} = \mathbf{0} & \text{on } \partial\Omega.\end{cases} \tag{5.1}$$

The variational formulation of (5.1) is to find a pair $(\boldsymbol{\psi}, \chi) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$ such that

$$a(\mathbf{v}, \boldsymbol{\psi}) + a_1(\mathbf{u}; \mathbf{v}, \boldsymbol{\psi}) + a_1(\mathbf{v}; \mathbf{u}, \boldsymbol{\psi}) - b(\mathbf{v}, \chi) = (\mathbf{u} - \mathbf{u}_h, \mathbf{v}), \quad \forall\, \mathbf{v} \in H_0^1(\Omega)^d, \tag{5.2a}$$

$$b(\boldsymbol{\psi}, q) = 0, \qquad \forall\, q \in L_0^2(\Omega). \tag{5.2b}$$

If $\mathbf{u}$ is a nonsingular solution of (1.1), then (5.2) has a unique solution [5].

To establish the error estimate in L^2 for the velocity, we use the duality argument introduced by Aubin and Nitsche [2]. To do so, we require that (5.1) be H^2-regular, i.e.,

$$\|\boldsymbol{\psi}\|_2 + \|\chi\|_1 \le C\, \|\mathbf{u} - \mathbf{u}_h\|. \tag{5.3}$$

We write out the argument in the Navier–Stokes case, since the Stokes case is covered by a somewhat simpler argument. Let $(\boldsymbol{\psi}, \chi)$ be the solution of (5.2) and let $(\boldsymbol{\psi}_h, \chi_h) \in \mathcal{NC}^h \times P^h$ satisfy

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{1,h} + \|\chi - \chi_h\| \le Ch(\|\boldsymbol{\psi}\|_2 + \|\chi\|_1). \tag{5.4}$$

Theorem 5.1 *Let* $(\mathbf{u}, p) \in H^2(\Omega)^2 \times H^1(\Omega)$ *and* $(\mathbf{u}_h, p_h) \in \mathcal{NC}^h \times P^h$ *be the solutions of* (1.2) *and* (2.5)*, respectively. If* (5.1) *is* H^2*-regular, then*

$$\|\mathbf{u} - \mathbf{u}_h\| \le Ch^2(\|\mathbf{u}\|_2 + \|p\|_1). \tag{5.5}$$

Proof The inequality (5.4) and the H^2-regularity (5.3) imply that

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{1,h} + \|\chi - \chi_h\| \le Ch\|\mathbf{u} - \mathbf{u}_h\|. \tag{5.6}$$

Multiplying both sides of the first equation of (5.1) by $\mathbf{u} - \mathbf{u}_h$, integrating by parts on each element, and using (2.3), we see that

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|^2 &= a_h(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\psi}) + a_{1,h}(\mathbf{u}; \mathbf{u} - \mathbf{u}_h, \boldsymbol{\psi}) \\
&\quad + a_{1,h}(\mathbf{u} - \mathbf{u}_h; \mathbf{u}, \boldsymbol{\psi}) - b_h(\mathbf{u} - \mathbf{u}_h, \chi) \\
&\quad - \nu \sum_j \left\langle \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{n}_j}, \mathbf{u} - \mathbf{u}_h \right\rangle_j - \sum_j \langle (\mathbf{u} \cdot \mathbf{n}_j)\boldsymbol{\psi}, \mathbf{u} - \mathbf{u}_h \rangle_j \\
&\quad + \sum_j \langle \chi, (\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{n}_j \rangle_j \\
&= a_h(\mathbf{u} - \mathbf{u}_h, \boldsymbol{\psi}) + a_{s,h}(\mathbf{u}; \mathbf{u} - \mathbf{u}_h, \boldsymbol{\psi}) + a_{s,h}(\mathbf{u} - \mathbf{u}_h; \mathbf{u}, \boldsymbol{\psi}) \\
&\quad - \frac{1}{2} \sum_j ((\nabla \cdot (\mathbf{u} - \mathbf{u}_h))\mathbf{u}, \boldsymbol{\psi})_j \\
&\quad - b_h(\mathbf{u} - \mathbf{u}_h, \chi) - \nu \sum_j \left\langle \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{n}_j}, \mathbf{u} - \mathbf{u}_h \right\rangle_j
\end{aligned}$$

$$- \frac{1}{2}\sum_j \langle (\mathbf{u}\cdot\mathbf{n}_j)\boldsymbol{\psi}, \mathbf{u}-\mathbf{u}_h\rangle_j + \frac{1}{2}\sum_j \langle ((\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}_j)\mathbf{u}, \boldsymbol{\psi}\rangle_j$$
$$+ \sum_j \langle \chi, (\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}_j\rangle_j. \tag{5.7}$$

The second equality above follows from (2.3) and the fact that $\mathbf{u}$ is divergence free. The difference of (4.19) and (2.5a), tested against $\mathbf{v} = \boldsymbol{\psi}_h$, implies that

$$\begin{aligned} 0 = {} & a_h(\mathbf{u}-\mathbf{u}_h, \boldsymbol{\psi}_h) + a_{s,h}(\mathbf{u}; \mathbf{u}, \boldsymbol{\psi}_h) \\ & - a_{s,h}(\mathbf{u}_h; \mathbf{u}_h, \boldsymbol{\psi}_h) - b_h(\boldsymbol{\psi}_h, p-p_h) \\ & - \nu \sum_j \left\langle \frac{\partial \mathbf{u}}{\partial \mathbf{n}_j}, \boldsymbol{\psi}_h \right\rangle_j + \frac{1}{2}\sum_j \langle (\mathbf{u}\cdot\mathbf{n}_j)\mathbf{u}, \boldsymbol{\psi}_h\rangle_j \\ & + \sum_j \langle p, \boldsymbol{\psi}_h\cdot\mathbf{n}_j\rangle_j. \end{aligned} \tag{5.8}$$

The difference of (5.7) and (5.8) gives

$$\begin{aligned} \|\mathbf{u}-\mathbf{u}_h\|^2 = {} & a_h(\mathbf{u}-\mathbf{u}_h, \boldsymbol{\psi}-\boldsymbol{\psi}_h) + \mathcal{R}_1 \\ & - b_h(\mathbf{u}-\mathbf{u}_h, \chi) + b_h(\boldsymbol{\psi}_h, p-p_h) + \mathcal{R}_2 + \mathcal{R}_3, \end{aligned} \tag{5.9}$$

where $\mathcal{R}_1$ is a sum of trilinear forms:

$$\begin{aligned} \mathcal{R}_1 = {} & a_{s,h}(\mathbf{u}; \mathbf{u}-\mathbf{u}_h, \boldsymbol{\psi}) + a_{s,h}(\mathbf{u}-\mathbf{u}_h; \mathbf{u}, \boldsymbol{\psi}) \\ & - a_{s,h}(\mathbf{u}; \mathbf{u}, \boldsymbol{\psi}_h) + a_{s,h}(\mathbf{u}_h; \mathbf{u}_h, \boldsymbol{\psi}_h) \\ = {} & a_{s,h}(\mathbf{u}; \mathbf{u}-\mathbf{u}_h, \boldsymbol{\psi}) + a_{s,h}(\mathbf{u}-\mathbf{u}_h; \mathbf{u}, \boldsymbol{\psi}) \\ & - a_{s,h}(\mathbf{u}; \mathbf{u}-\mathbf{u}_h, \boldsymbol{\psi}_h) - a_{s,h}(\mathbf{u}-\mathbf{u}_h; \mathbf{u}_h, \boldsymbol{\psi}_h) \\ = {} & a_{s,h}(\mathbf{u}; \mathbf{u}-\mathbf{u}_h, \boldsymbol{\psi}-\boldsymbol{\psi}_h) + a_{s,h}(\mathbf{u}-\mathbf{u}_h; \mathbf{u}, \boldsymbol{\psi}) \\ & - a_{s,h}(\mathbf{u}-\mathbf{u}_h; \mathbf{u}_h, \boldsymbol{\psi}) \\ = {} & a_{s,h}(\mathbf{u}; \mathbf{u}-\mathbf{u}_h, \boldsymbol{\psi}-\boldsymbol{\psi}_h) + a_{s,h}(\mathbf{u}-\mathbf{u}_h; \mathbf{u}-\mathbf{u}_h, \boldsymbol{\psi}) \\ & + a_{s,h}(\mathbf{u}-\mathbf{u}_h; \mathbf{u}_h, \boldsymbol{\psi}-\boldsymbol{\psi}_h); \end{aligned} \tag{5.10}$$

$\mathcal{R}_2$ is a sum of line integrals:

$$\begin{aligned} \mathcal{R}_2 = {} & -\nu \sum_j \left\langle \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{n}_j}, \mathbf{u}-\mathbf{u}_h \right\rangle_j - \frac{1}{2}\sum_j \langle (\mathbf{u}\cdot\mathbf{n}_j)\boldsymbol{\psi}, \mathbf{u}-\mathbf{u}_h\rangle_j \\ & + \frac{1}{2}\sum_j \langle ((\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}_j)\mathbf{u}, \boldsymbol{\psi}\rangle_j \\ & + \sum_j \langle \chi, (\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}_j\rangle_j + \nu \sum_j \left\langle \frac{\partial \mathbf{u}}{\partial \mathbf{n}_j}, \boldsymbol{\psi}_h \right\rangle_j \\ & - \frac{1}{2}\sum_j \langle (\mathbf{u}\cdot\mathbf{n}_j)\mathbf{u}, \boldsymbol{\psi}_h\rangle_j - \sum_j \langle p, \boldsymbol{\psi}_h\cdot\mathbf{n}_j\rangle_j; \end{aligned}$$

and $\mathcal{R}_3 = -\frac{1}{2}\sum_j((\nabla\cdot(\mathbf{u}-\mathbf{u}_h))\mathbf{u},\boldsymbol{\psi})_j$. Since $\mathbf{u},\boldsymbol{\psi}\in H_0^1(\Omega)^2 \hookrightarrow C^0(\Omega)^2$, we have

$$\begin{aligned}\mathcal{R}_2 = &-\nu\sum_j\left\langle\frac{\partial\boldsymbol{\psi}}{\partial\mathbf{n}_j},\mathbf{u}-\mathbf{u}_h\right\rangle_j - \frac{1}{2}\sum_j\langle(\mathbf{u}\cdot\mathbf{n}_j)\boldsymbol{\psi},\mathbf{u}-\mathbf{u}_h\rangle_j\\ &+\frac{1}{2}\sum_j\langle((\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}_j)\mathbf{u},\boldsymbol{\psi}\rangle_j\\ &+\sum_j\langle\chi,(\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{n}_j\rangle_j + \nu\sum_j\left\langle\frac{\partial\mathbf{u}}{\partial\mathbf{n}_j},\boldsymbol{\psi}_h-\boldsymbol{\psi}\right\rangle_j\\ &-\frac{1}{2}\sum_j\langle(\mathbf{u}\cdot\mathbf{n}_j)\mathbf{u},\boldsymbol{\psi}_h-\boldsymbol{\psi}\rangle_j - \sum_j\langle p,(\boldsymbol{\psi}_n-\boldsymbol{\psi})\cdot\mathbf{n}_j\rangle_j,\end{aligned}$$

which, together with Lemma 4.2, implies that

$$\begin{aligned}|\mathcal{R}_2| \le &\,Ch\,(\|\boldsymbol{\psi}\|_2+\|\chi\|_1)\,\|\mathbf{u}-\mathbf{u}_h\|_{1,h}\\ &+ Ch\,(\|\mathbf{u}\|_2+\|p\|_1)\,\|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{1,h}.\end{aligned}\tag{5.11}$$

Let c_j be a constant such that

$$\|\mathbf{u}\cdot\boldsymbol{\psi}-c_j\|_{\Omega_j} \le Ch\|\mathbf{u}\cdot\boldsymbol{\psi}\|_{1,\Omega_j} \le Ch\|\mathbf{u}\|_{1,\Omega_j}\|\boldsymbol{\psi}\|_{2,\Omega_j}.$$

Since $\nabla\cdot\mathbf{u}=0$ and $b_h(\mathbf{u}_h,q)=0$ for all $q\in P^h$,

$$\begin{aligned}|\mathcal{R}_3| &= \left|\frac{1}{2}\sum_j((\nabla\cdot(\mathbf{u}-\mathbf{u}_h))\mathbf{u},\boldsymbol{\psi})_j\right| = \left|\frac{1}{2}\sum_j(\nabla\cdot(\mathbf{u}-\mathbf{u}_h),\mathbf{u}\cdot\boldsymbol{\psi})_j\right|\\ &= \frac{1}{2}\left|\sum_j(\nabla\cdot(\mathbf{u}-\mathbf{u}_h),(\mathbf{u}\cdot\boldsymbol{\psi}-c_j))_j\right|\\ &\le Ch\|\mathbf{u}-\mathbf{u}_h\|_{1,h}\|\mathbf{u}\|_1\|\boldsymbol{\psi}\|_2.\end{aligned}\tag{5.12}$$

Combining (5.9)–(5.12) and using the Cauchy–Schwarz inequality elementwise, we see that

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|^2 \le C\big(&\|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{1,h}\|\mathbf{u}-\mathbf{u}_h\|_{1,h}\\ &+\|\mathbf{u}-\mathbf{u}_h\|_{1,h}\|\chi-\chi_h\| + \|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{1,h}\|p-p_h\|\\ &+||\mathbf{u}-\mathbf{u}_h||_{1,h}^2||\boldsymbol{\psi}||_1 + h\,(\|\boldsymbol{\psi}\|_2+\|\chi\|_1)\,\|\mathbf{u}-\mathbf{u}_h\|_{1,h}\\ &+h\,(\|\mathbf{u}\|_2+\|p\|_1)\,\|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{1,h}\big),\end{aligned}$$

which, together with (4.18), (5.3), and (5.6), implies the validity of (5.5). This completes the proof of the theorem. □

6 Two-dimensional quadrilateral elements

An extension to quadrilateral elements for the components of the velocity is immediate. If Ω_j is a quadrilateral, there is a unique (up to rotation in the order of the vertices) bilinear map $F_j : \hat{K} \to \Omega_j$ and F_j is affine on the edges of $\hat{K}$. Thus, if we define the basis on Ω_j as usual by

$$\mathcal{Q}(\Omega_j) = \{v : v = \hat{v} \circ F_j^{-1},\ \hat{v} \in \mathcal{Q}(\hat{K})\},$$

then the orthogonality properties (2.2) remain valid. Moreover, the two affine maps induced on a common edge between adjacent quadrilateral elements coincide, so that requiring continuity at midpoints of edges is consistent with the mappings. If shape quasiregularity is enforced on a partition into quadrilaterals, then the approximation properties (4.6) also remain valid. These properties allow us to observe that the entire convergence argument remains valid.

7 Three-dimensional rectangular elements

The results in the previous sections concerning rectangular elements can be extended to three dimensions without difficulty. Therefore, we limit ourselves to describing the nonconforming finite element approximation spaces, which are direct extensions of those in two dimensions. Thus, the pressure is approximated by piecewise constants and each component of the velocity by the nonconforming, three-dimensional elements $\mathcal{Q}$ defined below; again continuity is imposed at the midpoints of interelement faces, along with the requirement that the nodal values on the boundary vanish.

As in [4], the nonconforming three-dimensional element $\mathcal{Q}$ on the reference cube

$$\hat{K} = [-1, 1] \times [-1, 1] \times [-1, 1]$$

is chosen as

$$\begin{aligned}\mathcal{Q}(\hat{K}) = \ & \text{Span}\left\{1, \hat{x}_1, \hat{x}_2, \hat{x}_3, \left(\hat{x}_1^2 - \frac{5}{3}\hat{x}_1^4\right) - \left(\hat{x}_2^2 - \frac{5}{3}\hat{x}_2^4\right),\right. \\ & \left.\left(\hat{x}_1^2 - \frac{5}{3}\hat{x}_1^4\right) - \left(\hat{x}_3^2 - \frac{5}{3}\hat{x}_3^4\right)\right\} \\ = \ & \text{Span}\left\{1, \hat{x}_1, \hat{x}_2, \hat{x}_3, \left(\hat{x}_2^2 - \frac{5}{3}\hat{x}_2^4\right) - \left(\hat{x}_3^2 - \frac{5}{3}\hat{x}_3^4\right),\right. \\ & \left.\left(\hat{x}_2^2 - \frac{5}{3}\hat{x}_2^4\right) - \left(\hat{x}_1^2 - \frac{5}{3}\hat{x}_1^4\right)\right\}\end{aligned}$$

$$= \text{Span}\left\{1, \hat{x}_1, \hat{x}_2, \hat{x}_3, \left(\hat{x}_3^2 - \frac{5}{3}\hat{x}_3^4\right) - \left(\hat{x}_1^2 - \frac{5}{3}\hat{x}_1^4\right), \left(\hat{x}_3^2 - \frac{5}{3}\hat{x}_3^4\right) - \left(\hat{x}_2^2 - \frac{5}{3}\hat{x}_2^4\right)\right\}.$$

This choice again guarantees the orthogonality (6.1) of [4]. Denote the midpoints of faces and the associated faces of the reference element $\hat{K}$ by $\hat{a}_i,\ i = 1, \ldots, 6$, and $\hat{s}_i,\ i = 1, \ldots, 6$, respectively. Then, the nodal basis function related to $\hat{a}_1 = (1, 0, 0)$ is given by

$$\hat{\phi}_1(\hat{x}) = \frac{1}{6} + \frac{1}{2}\hat{x}_1 - \frac{1}{4}\left(\left(\hat{x}_1^2 - \frac{5}{3}\hat{x}_1^4\right) - \left(\hat{x}_2^2 - \frac{5}{3}\hat{x}_2^4\right)\right) - \frac{1}{4}\left(\left(\hat{x}_1^2 - \frac{5}{3}\hat{x}_1^4\right) - \left(\hat{x}_3^2 - \frac{5}{3}\hat{x}_3^4\right)\right);$$

the other five can be obtained by permuting indices and reflecting coordinates.

A direct manipulation verifies that, for $i, j = 1, \ldots, 6$,

$$\int_{\hat{s}_i} \hat{\phi}_j \, d\hat{s} = \delta_{ij} \left|\hat{s}_i\right|, \tag{7.1}$$

where $\left|\hat{s}_i\right|$ is the area of the face $\hat{s}_i$. The stability analysis and analyses of the errors $\mathbf{u} - \mathbf{u}_h$ and $p - p_h$ in Sect. 3 and Sect. 4 apply without modification.

References

[1] Brezzi, F., Fortin, M.: Mixed and hybrid finite element methods. New York: Springer 1991

[2] Ciarlet, P.G.: The finite element method for elliptic problems. Amsterdam: North-Holland 1978

[3] Crouzeix, M., Raviart, P.A.: Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I. Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge **7**, 33–76 (1973)

[4] Douglas, J., Jr., Santos, J.E., Sheen, D., Ye, X.: Nonconforming Galerkin methods based on quadrilateral elements for second order elliptic problems. RAIRO Modél. Math. Anal. Numér., to appear

[5] Girault, V., Raviart, P.A.: Finite element methods for Navier–Stokes equations. Theory and algorithms. Berlin: Springer 1986

[6] Han, H.-D.: Nonconforming elements in the mixed finite element method. J. Comput. Math. **2**, 223–233 (1984)

[7] Kellogg, R.B., Osborn, J.E.: A regularity result for the Stokes problem in a convex polygon. J. Funct. Anal. **21**, 397–431 (1976)

[8] Ladyzhenskaya, O.A.: The mathematical theory of viscous incompressible flow. New York: Gordon and Breach 1963

[9] Rannacher, R., Turek, S.: Simple nonconforming quadrilateral Stokes element. Numer. Methods Partial Differential Equations **8**, 97–111 (1992)

[10] Zine, A.: Méthodes hybrides pour les éléments finis non conformes, "Résolution d'un système de Stokes". Thèse, Univ. Paris 1984