Recovery-Based Error Estimators for Interface Problems: Mixed and Nonconforming Finite Elements (Extended Version)

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April 28, 2008

Abstract. In [13], we introduced and analyzed a recovery-based *a posteriori* error estimator for conforming linear finite element approximation to interface problems. It was shown theoretically that the estimator is robust with respect to the size of jumps provided that the distribution of coefficients is locally monotone. Numerical examples showed that this condition is unnecessary. This paper extends the idea in [13] to mixed and nonconforming finite element methods for developing and analyzing robust estimators. Numerical results on test problems are also presented.

1 Introduction

The recovery-based *a posteriori* error estimators have been extensively studied for conforming finite elements by many researchers due to their many appealing properties: simplicity, asymptotic exactness, and universality. The universality is in the sense that there is no need for the underlying residual or boundary value problem. For the mixed and nonconforming finite element methods, Carstensen and Bartels in [16] introduced and analyzed recovery-based error estimators. Their estimators for both the mixed and the nonconforming elements are based on the recovery of the gradient in $H^1(\Omega)^2$. These estimators work well for the Poisson equation even though the gradient of the exact solution only belongs to $H(\mathrm{div}) \cap H(\mathrm{curl})$ for non-convex polygonal domains. For other types of estimators on the mixed and the nonconforming methods, see [1, 2, 3, 5, 9, 15, 16, 17, 22, 23, 25, 28, 29, 30, 41, 42] and references therein.

As demonstrated numerically in [36, 37, 13] and theoretically in [14], for conforming finite element approximations to the interface problem with large jumps, existing estimators of the recovery type over-refine regions where there are no errors and, hence, fail to reduce the global error. This is also true for the recovery-based estimators in [16] for the mixed and nonconforming finite element methods (see Figures 1, 2, 7, and 8). The reason for the over-refinements is that the recovered gradient is continuous but the true gradient is

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This work was supported in part by the National Science Foundation under grants DMS-0511430 and DMS-0810855.

discontinuous. In other words, the over-refinements are caused by using continuous function to approximate discontinuous function in the recovery procedure. To overcome this difficulty, one often applies the method on each subdomain separately. For reasons why this local approach is not favorable, see detailed discussions in [37]. More importantly, the local approach fails when triangulations do not align with interfaces, which occurs when interfaces are curves/surfaces or have unknown locations. In [13], we introduced and analyzed a global approach for the conforming linear finite element approximation by recovering the flux in the $H(\text{div})$ conforming finite element spaces. The resulting estimator is then free of over-refinements and satisfies the efficiency and reliability bounds with constants independent of the size of jumps.

The purpose of this paper is to extend the idea in [13] to mixed and nonconforming finite element approximations. To do so, we need to determine what quantities to be recovered and which finite element spaces to be used. The guideline for such choices is based on our view that a recovery-based estimator is a measurement of the violation of finite element approximations on physical continuities. Therefore, the quantities to be recovered are those whose finite element approximations do not preserve the physical continuity. The interface problems in (2.1) have two physical continuities: the solution u and the normal component of the flux $\boldsymbol{\sigma} = -k\nabla u$. Mathematically, this means

$$u \in H^1(\Omega) \quad \text{and} \quad \boldsymbol{\sigma} \in H(\text{div}) \subset L^2(\Omega)^2. \tag{1.1}$$

For the mixed method, the continuity of the solution is violated while that of the flux is preserved. To measure such a violation, we recover the gradient of the solution. To choose proper finite element spaces, we notice that the first property in (1.1) implies

$$\nabla u \in H(\text{curl}). \tag{1.2}$$

Physically, the tangential components of vector fields in $H(\text{curl})$ are continuous. Therefore, the quantity to be recovered is the gradient and the proper finite element space is the $H(\text{curl})$ conforming finite element space. This choice accommodates discontinuity of the normal component of the gradient and, hence, eliminates over-refinements. For nonconforming finite element approximations, since both continuities are violated, we recover both the flux and the gradient in the $H(\text{div})$ and $H(\text{curl})$ conforming finite element spaces, respectively, through weighted L^2 projections. The estimator is then the average of two measurements: the weighted L^2 norms of differences between the direct and the recovered approximations of the flux and the gradient.

Estimators introduced in this paper are analyzed by establishing the standard reliability and efficiency bounds and are supported by numerical results. In particular, we prove theoretically that the estimators are robust, in the sense that the reliability and efficiency constants are independent of the size of jumps, provided that the distribution of coefficients is locally monotone. (In this paper, we will use C to denote a generic positive constant that is independent of the mesh parameter h_K and the size of jumps $k_{\max}/k_{\min}$ introduced in subsequent sections.) We also show numerically that there is no over-refinements along interfaces for a benchmark test problem whose coefficients are not locally monotone. Results in this paper may be extended to three-dimensions in a straightforward manner.

It is important to point out that research on robust estimators for interface problems is limited. For the conforming finite element method, robust *a posteriori* error estimators have been studied by Bernardi and Verfürth [8] and Petzoldt [38] for the residual-based estimator, Luce and Wohlmuth [31] for the equilibrated estimator, and us [13] for the recovery-based estimator. For the nonconforming elements, Ainsworth [2] studied a robust equilibrated estimator. For the mixed method, see a recent work by Ainsworth [3].

This paper is organized as follows. Section 2 introduces interface problems and variational formulations. Various finite element spaces and both mixed and nonconforming finite element approximations are described in section 3. Recovery procedures and *a posteriori* error estimators are defined in sections 4 and 5, respectively. Interpolation operators needed for analysis are introduced in section 6. We establish the efficiency and reliability bounds of estimators introduced in this paper in section 7. Finally, we present numerical results for test problems in section 8.

1.1 Function Spaces and Preliminaries

Let Ω be a bounded polygonal domain in $\Re^2$ with boundary $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. For a subdomain $G \subset \Omega$, we use the standard notations and definitions for the Sobolev spaces $H^s(G)$ and $H^s(\partial G)$ for $s \geq 0$. The standard associated inner products are denoted by $(\cdot,\, \cdot)_{s,G}$ and $(\cdot,\, \cdot)_{s,\partial G}$, and their respective norms are denoted by $\|\cdot\|_{s,G}$ and $\|\cdot\|_{s,\partial G}$. We omit the subscript G or ∂G if $G = \Omega$ from the inner product and norm designation when there is no risk of confusion.

In two dimensions, for a vector-valued function $\boldsymbol{\tau} = (\tau_1,\, \tau_2)^t$, define the divergence and curl operators by

$$\nabla\cdot\boldsymbol{\tau} := \frac{\partial \tau_1}{\partial x_1} + \frac{\partial \tau_2}{\partial x_2} \quad \text{and} \quad \nabla\times\boldsymbol{\tau} := \frac{\partial \tau_2}{\partial x_1} - \frac{\partial \tau_1}{\partial x_2},$$

respectively. For a scalar-valued function v, define the operator $\nabla^\perp$ by

$$\nabla^\perp v = Q\,\nabla v = (-\frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_1})^t \quad \text{with} \quad Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We shall use the following Hilbert spaces

$$H(\mathrm{div};\Omega) = \{\boldsymbol{\tau} \in L^2(\Omega)^2 \,:\, \nabla\cdot\boldsymbol{\tau} \in L^2(\Omega)\}$$
$$\text{and} \qquad H(\mathrm{curl};\Omega) = \{\boldsymbol{\tau} \in L^2(\Omega)^2 \,:\, \nabla\times\boldsymbol{\tau} \in L^2(\Omega)\}$$

equipped with the norms

$$\|\boldsymbol{\tau}\|_{H(\mathrm{div};\,\Omega)} = \left(\|\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\nabla\cdot\boldsymbol{\tau}\|_{0,\Omega}^2\right)^{\frac{1}{2}} \quad \text{and} \quad \|\boldsymbol{\tau}\|_{H(\mathrm{curl};\,\Omega)} = \left(\|\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\nabla\times\boldsymbol{\tau}\|_{0,\Omega}^2\right)^{\frac{1}{2}},$$

respectively. Denote their subspaces by

$$H_N(\mathrm{div};\Omega) = \{\boldsymbol{\tau} \in H(\mathrm{div};\Omega) \,:\, \boldsymbol{\tau}\cdot\mathbf{n}\big|_{\Gamma_N} = 0\}$$
$$\text{and} \qquad H_D(\mathrm{curl};\Omega) = \{\boldsymbol{\tau} \in H(\mathrm{curl};\Omega) \,:\, \boldsymbol{\tau}\cdot\mathbf{t}\big|_{\Gamma_D} = 0\},$$

where $\mathbf{n} = (n_1, n_2)^t$ and $\mathbf{t} = (t_1, t_2)^t = Q\,\mathbf{n} = (-n_2, n_1)^t$ are the unit vectors outward normal to and clockwise tangent to the boundary $\partial\Omega$, respectively. Finally, we will also use the following formula of integration by parts

$$(\nabla\times\boldsymbol{\tau},\, v) + (\boldsymbol{\tau},\, \nabla^{\perp} v) = \int_{\partial\Omega} (\boldsymbol{\tau}\cdot\mathbf{t})\, v\, ds \tag{1.3}$$

for all $\boldsymbol{\tau} \in H(\mathrm{curl}\,;\Omega)$ and all $v \in H^1(\Omega)$.

2 Interface Problems and Variational Forms

Consider the following interface problem

$$-\nabla\cdot\,(k(x)\nabla\, u) = f \quad \text{in } \Omega \tag{2.1}$$

with boundary conditions

$$u = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n}\cdot(k\nabla\, u) = 0 \quad \text{on } \Gamma_N, \tag{2.2}$$

where f is a given scalar-valued function in $L^2(\Omega)$, and $k(x)$ is positive and piecewise constant on polygonal subdomains of Ω with possible large jumps across subdomain boundaries (interfaces):

$$k(x) = k_i > 0 \quad \text{in } \Omega_i$$

for $i = 1, \ldots, n$. Here, $\{\Omega_i\}_{i=1}^n$ is a partition of the domain Ω with Ω_i being an open polygonal domain. Define

$$k_{\min} = \min_{1\le i\le n} k_i \quad \text{and} \quad k_{\max} = \max_{1\le i\le n} k_i.$$

For simplicity, we consider only homogeneous boundary conditions. Also, we assume that Γ_D is not empty (i.e., $\mathrm{mes}\,(\Gamma_D) \neq 0$).

Let

$$H_D^1(\Omega) := \{v \in H^1(\Omega)\,:\, v = 0 \text{ on } \Gamma_D\} \quad \text{and} \quad H_N^1(\Omega) := \{v \in H^1(\Omega)\,:\, v = 0 \text{ on } \Gamma_N\}.$$

The corresponding variational form of system (2.1) and (2.2) is to find $u \in H_D^1(\Omega)$ such that

$$a(u,\, v) = f(v) \quad \forall\, v \in H_D^1(\Omega), \tag{2.3}$$

where the bilinear and linear forms are defined by

$$a(u,\, v) = (k(x)\nabla u,\, \nabla v) \quad \text{and} \quad f(v) = (f,\, v),$$

respectively.

Introducing the flux defined by

$$\boldsymbol{\sigma} = -k(x)\nabla u \quad \text{in } \Omega,$$

then (2.1) may be rewritten as an equivalent first-order system:

$$\begin{cases} k^{-1}\boldsymbol{\sigma} + \nabla u &= 0 \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{\sigma} &= f \quad \text{in } \Omega \end{cases} \tag{2.4}$$

with boundary conditions

$$u = 0 \quad \text{on } \Gamma_D \quad \text{and} \quad \boldsymbol{\sigma} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N. \tag{2.5}$$

The corresponding mixed variational formulation is to find $(\boldsymbol{\sigma},\, u) \in H_N(\text{div};\Omega) \times L^2(\Omega)$ such that

$$\begin{cases} (k^{-1}\boldsymbol{\sigma},\, \boldsymbol{\tau}) - (\nabla\cdot\boldsymbol{\tau},\, u) &= 0 \qquad \forall\, \boldsymbol{\tau} \in H_N(\text{div};\Omega), \\ (\nabla\cdot\boldsymbol{\sigma},\, v) &= (f,\, v) \quad \forall\, v \in L^2(\Omega). \end{cases} \tag{2.6}$$

3 Finite Element Approximations

3.1 Finite Element Spaces

For simplicity of presentation, consider only triangular elements. Let $\mathcal{T} = \{K\}$ be a finite element partition of the domain Ω. Assume that the triangulation $\mathcal{T}$ is regular (see [20]); i.e., for all $K \in \mathcal{T}$, there exists a positive constant κ such that

$$h_K \le \kappa\, \rho_K,$$

where h_K denotes the diameter of the element K and ρ_K the diameter of the largest circle that may be inscribed in K. Note that the assumption of the regularity does not exclude highly, locally refined meshes. Furthermore, assume that interfaces

$$F = \{\partial\Omega_i \cap \partial\Omega_j \,|\, i, j = 1,\, ...,\, n\}$$

do not cut through any element $K \in \mathcal{T}$. (This assumption is needed for analysis and for explicit estimators, but not for implicit estimators introduced in this paper.)

Denote the set of all edges of the triangulation by

$$\mathcal{E} := \mathcal{E}_{\Omega} \cup \mathcal{E}_{D} \cup \mathcal{E}_{N},$$

where $\mathcal{E}_{\Omega}$ is the set of all interior element edges and $\mathcal{E}_{D}$ and $\mathcal{E}_{N}$ are the sets of all boundary edges belonging to the respective Γ_D and Γ_N. For each $e \in \mathcal{E}$, denote by m_e and h_e the midpoint and the length of the edge e, respectively; denote by $\mathbf{n}_e$ a unit vector normal to e. When $e \in \mathcal{E}_{D} \cup \mathcal{E}_{N}$, assume that $\mathbf{n}_e$ is the unit outward normal vector. For each interior edge $e \in \mathcal{E}_{\Omega}$, let K_e^+ and K_e^- be the two elements sharing the common edge e such that the unit outward normal vector of K_e^+ coincides with $\mathbf{n}_e$.

Let $P_k(K)$ be the space of polynomials of degree k on element K. Denote the conforming continuous piecewise linear finite element space and the Crouzeix-Raviart nonconforming piecewise linear finite element space [21] associated with the triangulation $\mathcal{T}$ by

$$\mathcal{U} = \{v \in H^1(\Omega) \,:\, v|_K \in P_1(K) \quad \forall\, K \in \mathcal{T}\}$$
$$\textit{and} \quad \mathcal{U}^{nc} = \{v \in L^2(\Omega) \,:\, v|_K \in P_1(K) \;\forall\, K \in \mathcal{T},\ \text{and } v \text{ is continuous at } m_e \;\forall\, e \in \mathcal{E}_\Omega\}$$

and their subspaces by

$$\mathcal{U}_D = \{v \in \mathcal{U} : \, v = 0 \text{ on } \Gamma_D\} \quad \text{and} \quad \mathcal{U}_D^{nc} = \{v \in \mathcal{U}^{nc} \, : \, v(m_e) = 0 \; \forall \; e \in \mathcal{E}_D\},$$

respectively.

Denote the local lowest order Raviart-Thomas [39, 12] and Brezzi-Douglas-Marini spaces [11, 12] on element $K \in \mathcal{T}$ by

$$RT_0(K) = P_0(K)^2 + \mathbf{x}\, P_0(K) \quad \text{and} \quad BDM_1(K) = P_1(K)^2,$$

respectively, where $\mathbf{x} = (x_1, x_2)$. Then the standard $H(\text{div};\Omega)$ conforming Raviart-Thomas and Brezzi-Douglas-Marini spaces are defined by

$$RT_0 = \{\boldsymbol{\tau} \in H_N(\text{div};\Omega) \, : \, \boldsymbol{\tau}|_K \in RT_0(K) \;\; \forall \; K \in \mathcal{T}\}$$

and

$$BDM_1 = \{\boldsymbol{\tau} \in H_N(\text{div};\Omega) \, : \, \boldsymbol{\tau}|_K \in BDM_1(K) \;\; \forall \; K \in \mathcal{T}\},$$

respectively. For convenience, denote RT_0 or BDM_1 by $\mathcal{V}_N$. Also, let

$$P_0 = \{v \in L^2(\Omega) \, : \, v|_K \in P_0(K) \; \forall \; K \in \mathcal{T}\}.$$

Denote the first and second types of local lowest order Nedelec spaces [34, 35] on element $K \in \mathcal{T}$ by

$$ND_1(K) = P_0(K)^2 + (x_2, -x_1)P_0(K) \quad \text{and} \quad ND_2(K) = P_1(K)^2.$$

Then the standard $H(\text{curl}\,;\Omega)$ conforming Nedelec spaces are defined by

$$ND_1 = \{\boldsymbol{\tau} \in H_D(\text{curl}\,;\Omega) \, : \, \boldsymbol{\tau}|_K \in ND_1(K) \;\; \forall \; K \in \mathcal{T}\}$$

and

$$ND_2 = \{\boldsymbol{\tau} \in H_D(\text{curl}\,;\Omega) \, : \, \boldsymbol{\tau}|_K \in ND_2(K) \;\; \forall \; K \in \mathcal{T}\},$$

respectively. For convenience, denote by $\mathcal{W}_D$ the Nedelec space ND_1 or ND_2.

Finally, we define the discrete gradient, divergence, and curl operators as follows:

$$(\nabla_h v)|_K := \nabla(v|_K), \quad (\nabla_h \cdot \boldsymbol{\tau})|_K := \nabla \cdot (\boldsymbol{\tau}|_K), \quad \text{and} \quad (\nabla_h \times \boldsymbol{\tau})|_K := \nabla\times(\boldsymbol{\tau}|_K)$$

for all $K \in \mathcal{T}$, respectively.

3.2 Finite element approximations

The mixed finite element method is to find $(\boldsymbol{\sigma}_m, u_m) \in \mathcal{V}_N \times P_0$ such that

$$\left\{ \begin{array}{rcll} (k^{-1}\boldsymbol{\sigma}_m,\, \boldsymbol{\tau}) - (\nabla\cdot\boldsymbol{\tau},\, u_m) & = & 0 & \forall\; \boldsymbol{\tau} \in \mathcal{V}_N, \\ (\nabla\cdot\boldsymbol{\sigma}_m,\, v) & = & (f,\, v) & \forall\; v \in P_0. \end{array} \right. \tag{3.1}$$

Let $(\boldsymbol{\sigma}, u)$ and $(\boldsymbol{\sigma}_m, u_m)$ be the solutions of (2.6) and (3.1), respectively, and denote the true error of the mixed finite element approximation by

$$(E_m, e_m) = (\boldsymbol{\sigma} - \boldsymbol{\sigma}_m, u - u_m). \tag{3.2}$$

Then the difference between (2.6) and (3.1) gives the following error equations

$$\left\{ \begin{array}{rcll} (k^{-1}E_m, \boldsymbol{\tau}) - (\nabla \cdot \boldsymbol{\tau}, e_m) & = & 0 & \forall\, \boldsymbol{\tau} \in \mathcal{V}_N, \\ (\nabla \cdot E_m, v) & = & 0 & \forall\, v \in P_0. \end{array} \right. \tag{3.3}$$

Let $Q_h : L^2(\Omega) \mapsto P_0$ be the L^2 projection, then the second equation in (3.1) gives

$$\nabla \cdot \boldsymbol{\sigma}_m = Q_h f = f_h \in P_0 \tag{3.4}$$

with

$$f_h|_K = \frac{1}{|K|} \int_K f\, dx \quad \forall\, K \in \mathcal{T}.$$

Let $\Pi_h : H(\mathrm{div}; \Omega) \cap L^t(\Omega)^2 \mapsto \mathcal{V}_N$ for fixed $t > 2$ be the well-known RT/BDM interpolation operator which satisfies the commutativity property:

$$\nabla \cdot (\Pi_h \boldsymbol{\tau}) = Q_h \nabla \cdot \boldsymbol{\tau} \qquad \forall\, \boldsymbol{\tau} \in H(\mathrm{div}; \Omega) \cap L^t(\Omega)^d$$

and the approximation property for $\boldsymbol{\tau} \in H^s(\Omega)^2$:

$$\|\boldsymbol{\tau} - \Pi_h \boldsymbol{\tau}\|_{0,\Omega_i} \le C \left(\sum_{K \in \mathcal{T} \cap \Omega_i} h_K^{2s} \|\boldsymbol{\tau}\|_{s,K}^2 \right)^{1/2} \quad \text{for } 0 \le s \le 1 \text{ and } i = 1, ..., n.$$

Here and thereafter, we use C with or without subscripts in this paper to denote a generic positive constant, possibly different at different occurrences, that is independent of the mesh parameter h_K and the ratio $k_{\max}/k_{\min}$ but may depend on the domain Ω.

Theorem 3.1. *Assume that the solution, $(\boldsymbol{\sigma}, u)$, of problem (2.6) belongs to $H^s(\Omega) \times H^{1+s}(\Omega)$ with $0 \le s \le 1$. Then we have the following a priori error bound:*

$$\|k^{-1/2} E_m\|_{0,\Omega} \le C\, \|h^s k^{1/2} \nabla u\|_{s,\Omega} \tag{3.5}$$

with $\|h^s k^{1/2} \nabla u\|_{s,\Omega} = \left(\sum_{K \in \mathcal{T}} h_K^{2s}\, \|k^{1/2} \nabla u\|_{s,K}^2 \right)^{1/2}$.

Proof. By the commutativity property and (3.4), we have

$$\nabla \cdot (\Pi_h \boldsymbol{\sigma}) = Q_h \nabla \cdot \boldsymbol{\sigma} = Q_h f = \nabla \cdot \boldsymbol{\sigma}_m,$$

which, together with the first equation in (3.3) and the Cauchy-Schwarz inequality, implies

$$\begin{aligned} \|k^{-1/2} E_m\|_{0,\Omega}^2 &= (k^{-1} E_m, \boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}) + (k^{-1} E_m, \Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_m) \\ &= (k^{-1} E_m, \boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}) + (\nabla \cdot (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_m), e_m) \\ &= (k^{-1} E_m, \boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}) \le \|k^{-1/2} E_m\|_{0,\Omega}\, \|k^{-1/2} (\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma})\|_{0,\Omega}. \end{aligned}$$

Now, (3.5) is a direct consequence of the approximation property and $\boldsymbol{\sigma} = -k \nabla u$. □

The nonconforming finite element method is to find $u_{nc} \in \mathcal{U}_D^{nc}$ such that

$$(k\nabla_h u_{nc},\, \nabla_h v) \;=\; (f, v) \qquad \forall\, v \in \mathcal{U}_D^{nc}. \tag{3.6}$$

Let u and u_{nc} be the solutions of (2.3) and (3.6), respectively, and denote the true error of the nonconforming finite element approximation by

$$e_{nc} = u - u_{nc}. \tag{3.7}$$

Since $\mathcal{U}_D \subset \mathcal{U}_D^{nc}$, then we have the following error equation:

$$(k\nabla_h e_{nc},\, \nabla v) = (k(\nabla u - \nabla_h u_{nc}),\, \nabla v) = 0 \quad \forall\; v \in \mathcal{U}_D. \tag{3.8}$$

Lemma 3.2. *Let v_c be the orthogonal projection of $v \in H_D^1(\Omega) \cap H^{1+s}(\Omega)$ with $0 \le s \le 1$ onto $\mathcal{U}_D$ with respect to the inner product $(k\nabla\cdot, \nabla\cdot)$, then*

$$\|k^{1/2}\nabla(v - v_c)\|_{0,\Omega} \le C \left(\sum_{K\in\mathcal{T}} h_K^{2s}\, \|k^{1/2}\nabla v\|_{s,K}^2 \right)^{1/2}.$$

Proof. This can be proved similarly as that of Proposition 2.4 in [8]. □

For and $K \in \mathcal{T}$ and any $v \in H^{1+s}(K)$, $0 < s \le 1$, denote by

$$B_{s,K}(h_K,\, v) = \begin{cases} h_K^s \|k^{1/2}\nabla v\|_{s,K} & s \in (1/2,\, 1], \\[2ex] h_K^s \|k^{1/2}\nabla v\|_{s,K} + h_K k_K^{-1/2} \|f\|_{0,K} & s \in (0,\, 1/2]. \end{cases}$$

Lemma 3.3. *For any $w \in \mathcal{U}_D^{nc}$ and any $K \in \mathcal{T}$, let $\bar{w}_{K,e}$ be the mean value of $w|_K$ over edge $e \in \partial K$. Assume that the solution u of problem (2.3) belongs to $H^{1+s}(\Omega)$ with $s \in (0,\, 1]$, then*

$$\sum_{e\in\partial K} \Big| \int_e (\mathbf{n}\cdot k\nabla u)\,(w - \bar{w}_{K,e})\, ds \Big| \le C\, B_{s,K}(h_K,\, u)\, \|\nabla w\|_{0,K}. \tag{3.9}$$

For $v \in H^{1+s}(K)$, note that integral $\int_e (\mathbf{n}\cdot k\nabla v)\, w ds$ is the standard integration in $L^2(e)$ if $s > 1/2$. When $s \in (0, 1/2]$, it should be viewed as the duality pairing $\langle(\mathbf{n}\cdot k\nabla v), w\rangle_e$, where $(\mathbf{n}\cdot k\nabla v)|_e \in H^{\epsilon-1/2}(e)$ and $w|_e \in H^{1/2-\epsilon}(e)$ for any positive $\epsilon < s$.

Proof. The definition of $\bar{w}_{K,e}$ implies

$$\int_e (\mathbf{n}\cdot k\nabla u)\,(w - \bar{w}_{K,e})\, ds = \int_e (\mathbf{n}\cdot k\nabla u - \zeta_e)\,(w - \bar{w}_{K,e})\, ds, \tag{3.10}$$

for any constant ζ_e. When $s \in (1/2,\, 1]$, let $\zeta_e = h_e^{-1} \int_e \mathbf{n}\cdot k\nabla u\, ds$. It then follows from (3.10), the Cauchy-Schwarz inequality, the approximation property, and the trace theorem that

$$\begin{aligned} &\Big| \int_e (\mathbf{n}\cdot k\nabla u)\,(w - \bar{w}_{K,e})\, ds \Big| \le \|\mathbf{n}\cdot k\nabla u - \zeta_e\|_{0,e} \|w - \bar{w}_{K,e}\|_{0,e} \\ \le\;& C\, h_e^s \|\mathbf{n}\cdot k\nabla u\|_{s-1/2,e} \|\nabla w\|_{0,K} \le C\, h_K^s \|k^{1/2}\nabla v\|_{s,K} \|k^{1/2}\nabla w\|_{0,K}, \end{aligned}$$

which implies (3.9) for $s \in (1/2,\, 1]$.

When $s \in (0,\, 1/2]$, using (3.10), the definition of the dual norm, and the approximation property, we have

$$\begin{aligned} \Big| \int_e (\mathbf{n} \cdot k\nabla u)\,(w - \bar{w}_{K,e})\,ds \Big| &\le \|\mathbf{n} \cdot k\nabla u - \zeta_e\|_{-1/2+\epsilon,e} \|w - \bar{w}_{K,e}\|_{1/2-\epsilon,e} \\ &\le C\, h^{\epsilon} \|\mathbf{n} \cdot k\nabla u - \zeta_e\|_{-1/2+\epsilon,e} \|\nabla w\|_{0,K}. \end{aligned}$$

Let $\bar{\boldsymbol{\zeta}}$ be the mean value of $k\nabla u$ over K. Choosing $\zeta_e = \bar{\boldsymbol{\zeta}} \cdot \mathbf{n}_e$ and using (2.1) and the fact [7] that $\|\nabla \phi \cdot \mathbf{n}\|_{-1/2+\epsilon,e} \le C\, \left(\|\nabla \phi\|_{\epsilon,K} + h_K^{1-\epsilon} \|\Delta \phi\|_{0,K} \right)$ for any $\phi \in H^{1+\epsilon}(K)$ with $\Delta \phi \in L^2(K)$ and for any $0 < \epsilon < 1/2$, we have

$$\begin{aligned} & \|\mathbf{n} \cdot k\nabla u - \zeta_e\|_{-1/2+\epsilon,e} = \|\mathbf{n} \cdot (k\nabla u - \boldsymbol{\zeta})\|_{-1/2+\epsilon,e} \\ \le\;& C\, \left(\|k\nabla u - \bar{\boldsymbol{\zeta}}\|_{\epsilon,K} + h_K^{1-\epsilon} \|f\|_{0,K} \right) \le C\, \left(h_K^{s-\epsilon} \|k\nabla u\|_{s,K} + h_K^{1-\epsilon} \|f\|_{0,K} \right). \end{aligned}$$

Combining the above two inequalities yields (3.9) for $s \in (0,\, 1/2]$. This completes the proof of the lemma. □

Theorem 3.4. *Assume that the solution u of problem (2.3) belongs to $H^{1+s}(\Omega)$ with $0 < s \le 1$. Then we have the following a priori error bound*

$$\|k^{1/2} \nabla_h (u - u_{nc})\|_{0,\Omega} \le C \left(\sum_{K \in \mathcal{T}} B_{s,K}^2(h_K,\, u) \right)^{1/2}.$$

Proof. Since $\mathcal{U}_D \subset \mathcal{U}_D^{nc}$, Lemma 3.2 with $v = u$ gives

$$\inf_{v \in \mathcal{U}_D^{nc}} \|k^{1/2} \nabla_h (u - v)\|_{0,\Omega} \le \|k^{1/2} \nabla_h (u - u_c)\|_{0,\Omega} \le C \left(\sum_{K \in \mathcal{T}} h_K^{2s} \|k^{1/2} \nabla u\|_{s,K}^2 \right)^{1/2}.$$

For any $w \in \mathcal{U}_D^{nc}$ and any $K \in \mathcal{T}$, let $\bar{w}_{K,e}$ be the mean value of $w|_K$ over $e \in \partial K$. Let K^- be the element sharing the common edge e, then the continuity of w at the midpoint of e implies that $\bar{w}_{K,e} = \bar{w}_{K^-,e}$. For $e \in \mathcal{E}_D$, $w(m_e) = 0$ implies that $\bar{w}_{K,e} = 0$. Then it follows from integration by parts, (2.1), and the continuity of $\mathbf{n} \cdot k\nabla u$ across edge e that

$$(k\nabla u,\, \nabla_h w) - (f,\, w) = \sum_{K \in \mathcal{T}} \int_{\partial K} (\mathbf{n} \cdot k\nabla u)\, w\, ds = \sum_{K \in \mathcal{T}} \sum_{e \in \partial K} \int_e (\mathbf{n} \cdot k\nabla u)\,(w - \bar{w}_{K,e})\, ds,$$

which, together with the triangle inequality, Lemma 3.3, and the Cauchy-Schwarz inequality, give

$$\Big| (k\nabla u,\, \nabla_h w) - (f,\, w) \Big| \le C \left(\sum_{K \in \mathcal{T}} B_{s,K}^2(h_K,\, u) \right)^{1/2} \|k^{1/2} \nabla_h w\|_{0,\Omega}.$$

Now, Theorem 3.4 is a direct consequence of Strang's Lemma (e.g., [20]). □

4 Gradient and/or Flux Recovery

4.1 Implicit Approximation

For the mixed finite element approximation $(\boldsymbol{\sigma}_m, u_m)$, the continuity of the solution and, hence, the continuity of the tangential component of the gradient are violated while that of the flux is preserved. This suggests to recover the gradient in the $H(\text{curl})$ conforming finite element space ND_2. Since

$$\nabla u = -k^{-1}\boldsymbol{\sigma}, \tag{4.1}$$

we recover the gradient by solving the following variational problem: find $\boldsymbol{\rho}_m \in ND_2$ such that

$$(k\,\boldsymbol{\rho}_m,\, \boldsymbol{\tau}) = -(\boldsymbol{\sigma}_m,\, \boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in ND_2. \tag{4.2}$$

Theorem 4.1. *There exists a positive constant C independent of the ratio $k_{\max}/k_{\min}$ such that the following a priori error bound*

$$\|k^{1/2}(\nabla u - \boldsymbol{\rho}_m)\|_{0,\Omega} \le C \left(\inf_{\boldsymbol{\tau}\in ND_2} \|k^{1/2}(\nabla u - \boldsymbol{\tau})\|_{0,\Omega} + \|k^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_m)\|_{0,\Omega} \right) \tag{4.3}$$

holds.

Proof. (4.1) and (4.2) give the following error equation

$$(k\,(\nabla u - \boldsymbol{\rho}_m),\, \boldsymbol{\tau}) = (\boldsymbol{\sigma}_m - \boldsymbol{\sigma},\, \boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in ND_2,$$

which, together with the Cauchy-Schwarz, the triangle, and the arithmetic and geometric means inequalities, implies

$$\begin{aligned}
&\|k^{1/2}(\nabla u - \boldsymbol{\rho}_m)\|_{0,\Omega}^2 = (k(\nabla u - \boldsymbol{\rho}_m),\, \nabla u - \boldsymbol{\tau}) + (k(\nabla u - \boldsymbol{\rho}_m),\, \boldsymbol{\tau} - \boldsymbol{\rho}_m)\\
=\;& (k(\nabla u - \boldsymbol{\rho}_m),\, \nabla u - \boldsymbol{\tau}) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}_m,\, \boldsymbol{\tau} - \nabla u) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}_m,\, \nabla u - \boldsymbol{\rho}_m)\\
\le\;& \|k^{1/2}(\nabla u - \boldsymbol{\rho}_m)\|_{0,\Omega} \left(\|k^{1/2}(\nabla u - \boldsymbol{\tau})\|_{0,\Omega} + \|k^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_m)\|_{0,\Omega} \right)\\
&+ \|k^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_m)\|_{0,\Omega}\, \|k^{1/2}(\nabla u - \boldsymbol{\tau})\|_{0,\Omega}\\
\le\;& \frac{1}{2}\|k^{1/2}(\nabla u - \boldsymbol{\rho}_m)\|_{0,\Omega}^2 + C \left(\|k^{1/2}(\nabla u - \boldsymbol{\tau})\|_{0,\Omega}^2 + \|k^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_m)\|_{0,\Omega}^2 \right)^{1/2}.
\end{aligned}$$

This yields (4.3) and, hence, completes the proof of the theorem. □

For the nonconforming finite element approximation u_{nc}, both the continuities of the tangential component of the gradient and the normal component of the flux are violated. Hence, we recover both the gradient and flux as follows: finding $\boldsymbol{\rho}_{nc} \in \mathcal{W}_D$ such that

$$(k\boldsymbol{\rho}_{nc},\, \boldsymbol{\tau}) = (k\nabla_h u_{nc},\, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathcal{W}_D \tag{4.4}$$

and finding $\boldsymbol{\sigma}_{nc} \in \mathcal{V}_N$ such that

$$(k^{-1}\boldsymbol{\sigma}_{nc},\, \boldsymbol{\tau}) = -(\nabla_h u_{nc},\, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathcal{V}_N. \tag{4.5}$$

Theorem 4.2. *There exists a positive constant C independent of the ratio $k_{\max}/k_{\min}$ such that the following a priori error bounds*

$$\|k^{1/2}(\nabla u - \boldsymbol{\rho}_{nc})\|_{0,\Omega} \le C \left(\inf_{\boldsymbol{\tau}\in\mathcal{W}_D} \|k^{1/2}(\nabla u - \boldsymbol{\tau})\|_{0,\Omega} + \|k^{1/2}(\nabla u - \nabla_h u_{nc})\|_{0,\Omega} \right) \quad (4.6)$$

and

$$\|k^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{nc})\|_{0,\Omega} \le C \left(\inf_{\boldsymbol{\tau}\in\mathcal{V}_N} \|k^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\tau})\|_{0,\Omega} + \|k^{1/2}(\nabla u - \nabla_h u_{nc})\|_{0,\Omega} \right) \quad (4.7)$$

hold.

Proof. (4.6) and (4.7) may be proved in a similar fashion as that of Theorem 4.1 with the error equations:

$$(k(\nabla u - \boldsymbol{\rho}_{nc}),\, \boldsymbol{\tau}) = (k(\nabla u - \nabla_h u_{nc}),\, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathcal{W}_D$$

and

$$(k^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{nc}),\, \boldsymbol{\tau}) = (\nabla_h u_{nc} - \nabla u,\, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathcal{V}_N,$$

respectively. □

4.2 Explicit Approximations

Let $\delta_{e\,e'}$ denote the Kronecker delta:

$$\delta_{e\,e'} = \begin{cases} 1, & \text{if } e = e', \\ 0, & \text{if } e \neq e'. \end{cases}$$

Nodal basis functions of RT_0, BDM_1, ND_1, and ND_2 corresponding to the edge $e \in \mathcal{E}$ are characterized as follows:

(1) For RT_0, $\boldsymbol{\phi}_e$ is uniquely determined by

$$\int_{e'} \boldsymbol{\phi}_e \cdot \mathbf{n}_{e'}\, ds = \delta_{e\,e'} \quad \forall\, e' \in \mathcal{E};$$

Notice that $\boldsymbol{\phi}_e|_{e'} \cdot \mathbf{n}_{e'}$ is a constant and equals to $\dfrac{\delta_{ee'}}{|e|}$.

(2) For BDM_1, $\boldsymbol{\phi}_{e,i}$ $(i = 1, 2)$ are uniquely determined by

$$\int_{e'} \boldsymbol{\phi}_{e,1} \cdot \mathbf{n}_{e'}\, ds = \delta_{e\,e'} \quad \text{and} \quad \int_{e'} s\boldsymbol{\phi}_{e,1} \cdot \mathbf{n}_{e'}\, ds = 0 \quad \forall\, e' \in \mathcal{E},$$

$$\int_{e'} \boldsymbol{\phi}_{e,2} \cdot \mathbf{n}_{e'}\, ds = 0 \quad \text{and} \quad \int_{e'} s\boldsymbol{\phi}_{e,2} \cdot \mathbf{n}_{e'} ds = \delta_{e\,e'} \quad \forall\, e' \in \mathcal{E},$$

where s is a local coordinate on e' ranging from -1 to 1;

Notice that $\boldsymbol{\phi}_{e,1}$ of BDM_1 and $\boldsymbol{\phi}$ of RT_0 are the same. Since $\boldsymbol{\phi}_{e,1}|_e \cdot \mathbf{n}_e = 1/|e|$, we have the following orthogonality property:

$$\int_e (\boldsymbol{\phi}_{e,1} \cdot \mathbf{n}_e)(\boldsymbol{\phi}_{e,2} \cdot \mathbf{n}_e) ds = \frac{1}{|e|} \int_e (\boldsymbol{\phi}_{e,2} \cdot \mathbf{n}_e) ds = 0.$$

(3) For ND_1, $\boldsymbol{\psi}_e$ is uniquely determined by

$$\int_{e'} \boldsymbol{\psi}_e \cdot \mathbf{t}_{e'}\, ds = \delta_{e\,e'} \quad \forall\ e' \in \mathcal{E};$$

Similarly, $\boldsymbol{\psi}_e|_{e'} \cdot \mathbf{t}_{e'} = \dfrac{\delta_{ee'}}{|e|}$.

(4) For ND_2, $\boldsymbol{\psi}_{e,i}$ $(i = 1, 2)$ are uniquely determined by

$$\int_{e'} \boldsymbol{\psi}_{e,1} \cdot \mathbf{t}_{e'} ds = \delta_{e\,e'} \qquad \text{and} \qquad \int_{e'} s\boldsymbol{\psi}_{e,1} \cdot \mathbf{t}_{e'} ds = 0 \quad \forall\ e' \in \mathcal{E},$$

$$\int_{e'} \boldsymbol{\psi}_{e,2} \cdot \mathbf{t}_{e'} ds = 0 \qquad \text{and} \qquad \int_{e'} s\boldsymbol{\psi}_{e,2} \cdot \mathbf{t}_{e'} ds = \delta_{e\,e'} \quad \forall\ e' \in \mathcal{E}.$$

Similarly, $\boldsymbol{\psi}_{e,1}$ of ND_2 and $\boldsymbol{\psi}$ of ND_1 are the same. The following orthogonality property holds:

$$\int_e (\boldsymbol{\psi}_{e,1} \cdot \mathbf{t}_e)(\boldsymbol{\psi}_{e,2} \cdot \mathbf{t}_e) ds = 0. \tag{4.8}$$

Lemma 4.1. *For any element $K \in \mathcal{T}$, every constant vector $\boldsymbol{\tau}$ on K has the following representations in RT_0 and ND_1:*

$$\boldsymbol{\tau} = \sum_{e \in \partial K} \tau_e \boldsymbol{\phi}_e \quad \textit{with} \quad \tau_e = \int_e (\boldsymbol{\tau} \cdot \mathbf{n}_e)\, ds$$

and

$$\boldsymbol{\tau} = \sum_{e \in \partial K} \tau_e\, \boldsymbol{\psi}_e \quad \textit{with} \quad \tau_e = \int_e (\boldsymbol{\tau} \cdot \mathbf{t}_e)\, ds,$$

respectively; Every linear vector $\boldsymbol{\tau}$ on K has the following representations in BDM_1 and ND_2:

$$\boldsymbol{\tau} = \sum_{e \in \partial K} (\tau_{e,1}\, \boldsymbol{\phi}_{e,1} + \tau_{e,2}\, \boldsymbol{\phi}_{e,2}) \quad \textit{with} \quad \tau_{e,i} = \int_e s^{i-1} (\boldsymbol{\tau} \cdot \mathbf{n}_e)\, ds$$

and

$$\boldsymbol{\tau} = \sum_{e \in \partial K} (\tau_{e,1}\, \boldsymbol{\psi}_{e,1} + \tau_{e,2}\, \boldsymbol{\psi}_{e,2}) \quad \textit{with} \quad \tau_{e,i} = \int_e s^{i-1} (\boldsymbol{\tau} \cdot \mathbf{t}_e)\, ds,$$

respectively.

Proof. The lemma is a direct consequence of the fact, that both $RT_0(K)$ and $ND_1(K)$ and both $BDM_1(K)$ and $ND_2(K)$ contain the respective constant and linear vectors, and the characteristic equations for nodal basis functions. □

Lemma 4.2. *For a linear function v on edge e, let $\{\boldsymbol{\psi}_{e,i}\}_{i=1}^2$ be the ND_2 basis functions on e, then*

$$v = v_1\, \boldsymbol{\psi}_{e,1} \cdot \mathbf{t}_e + v_2\, \boldsymbol{\psi}_{e,2} \cdot \mathbf{t}_e \quad \textit{with} \quad v_i = \int_e s^{i-1} v\, ds \tag{4.9}$$

and

$$\int_e |v|^2\, ds = v_1^2 \int_e |\boldsymbol{\psi}_{e,1}|^2 ds + v_2^2 \int_e |\boldsymbol{\psi}_{e,2}|^2\, ds. \tag{4.10}$$

Proof. By the fact that $\boldsymbol{\psi}_{e,i} \cdot \mathbf{t}_e$ span the linear function space on e and the characteristic properties of $\boldsymbol{\psi}_{e,i}$, we have (4.9). (4.10) is the result of (4.9) and (4.8). $\square$

Now we are ready to introduce explicit approximations to the gradient based on the mixed finite element solution, and the flux and the gradient based on the nonconforming finite element solution.

For $\boldsymbol{\sigma}_m \in \mathcal{V}_N$, let $\boldsymbol{\tau}_1 = -k^{-1}\boldsymbol{\sigma}_m$ which is a linear vector-valued function on each K, so we first define its approximation $\hat{\boldsymbol{\rho}}_m \in ND_2$ by

$$\hat{\boldsymbol{\rho}}_m(\boldsymbol{\sigma}_m) = \sum_{e\in\mathcal{E}} \left(\hat{\rho}_{e,1}\boldsymbol{\psi}_{e,1} + \hat{\rho}_{e,2}\boldsymbol{\psi}_{e,2}\right), \tag{4.11}$$

where $\hat{\rho}_{e,1}$ and $\hat{\rho}_{e,2}$ are the zero- and one-moments of the tangential component of $\hat{\boldsymbol{\rho}}_m$ on the edge $e \in \mathcal{E}_\Omega \cup \mathcal{E}_D$, respectively. For $i = 1,\, 2$, $\hat{\rho}_{e,i}$ is defined by

$$\hat{\rho}_{e,i} := \begin{cases} \gamma_{1,e} \int_e s^{i-1}(\boldsymbol{\tau}_1|_{K_e^+} \cdot \mathbf{t}_e)\, ds + (1-\gamma_{1,e}) \int_e s^{i-1}(\boldsymbol{\tau}_1|_{K_e^-} \cdot \mathbf{t}_e)\, ds & \text{for } e \in \mathcal{E}_\Omega, \\ \int_e s^{i-1}(\boldsymbol{\tau}_1 \cdot \mathbf{t}_e)\, ds & \text{for } e \in \mathcal{E}_D \end{cases} \tag{4.12}$$

for some constant $\gamma_{1,e} \in [0,\, 1]$.

Let $\boldsymbol{\tau}_2 = -k\nabla_h u_{nc}$. Since $\boldsymbol{\tau}_2$ is piecewise constant, then it suffices to approximate it using RT_0. Define the explicit approximation $\hat{\boldsymbol{\sigma}}_{nc}(u_{nc})$ in $RT_0 = \text{span}\{\boldsymbol{\phi}_e : e \in \mathcal{E}\}$ by

$$\hat{\boldsymbol{\sigma}}_{nc}(u_{nc}) = \sum_{e\in\mathcal{E}} \hat{\sigma}_e\, \boldsymbol{\phi}_e, \tag{4.13}$$

where $\hat{\sigma}_e$ is the normal component of $\hat{\boldsymbol{\sigma}}_{nc}(u_{nc})$ on the edge $e \in \mathcal{E}$ defined by

$$\hat{\sigma}_e := \begin{cases} \gamma_{2,e} \int_e (\boldsymbol{\tau}_2|_{K_e^+} \cdot \mathbf{n}_e)\, ds + (1-\gamma_{2,e}) \int_e (\boldsymbol{\tau}_2|_{K_e^-} \cdot \mathbf{n}_e)\, ds & \text{for } e \in \mathcal{E}_\Omega, \\ \int_e (\boldsymbol{\tau}_2|_e \cdot \mathbf{n}_e)\, ds & \text{for } e \in \mathcal{E}_D \end{cases} \tag{4.14}$$

for some constant $\gamma_{2,e} \in [0,\, 1]$. To ensure the efficiency bound independent of the size of jumps, we choose

$$\gamma_{2,e} = \frac{\sqrt{k_{K_e^-}}}{\sqrt{k_{K_e^+}} + \sqrt{k_{K_e^-}}}. \tag{4.15}$$

Let $\boldsymbol{\tau}_3 = \nabla_h u_{nc}$. Since $\boldsymbol{\tau}_3$ is piecewise constant, we approximate it in $ND_1 = \text{span}\{\boldsymbol{\psi}_e : e \in \mathcal{E}\}$ by

$$\hat{\boldsymbol{\rho}}_{nc}(u_{nc}) = \sum_{e\in\mathcal{E}} \hat{\rho}_e\, \boldsymbol{\psi}_e, \tag{4.16}$$

where $\hat{\rho}_e$ is the tangential component of $\hat{\boldsymbol{\rho}}_{nc}(u_{nc})$ on the edge $e \in \mathcal{E}$ defined by

$$\hat{\rho}_e := \begin{cases} \gamma_{3,e} \int_e (\boldsymbol{\tau}_3|_{K_e^+} \cdot \mathbf{t}_e)\, ds + (1-\gamma_{3,e}) \int_e (\boldsymbol{\tau}_3|_{K_e^-} \cdot \mathbf{t}_e)\, ds & \text{for } e \in \mathcal{E}_\Omega, \\ \int_e (\boldsymbol{\tau}_3 \cdot \mathbf{t}_e) ds & \text{for } e \in \mathcal{E}_N \end{cases} \tag{4.17}$$

for some constant $\gamma_{3,e} \in [0, 1]$. To ensure the efficiency bound independent of the size of jumps, we choose

$$\gamma_{3,e} = \frac{\sqrt{k_{K_e^+}}}{\sqrt{k_{K_e^+}} + \sqrt{k_{K_e^-}}}. \tag{4.18}$$

5 A Posteriori Error Estimators

5.1 Error estimators for mixed elements

For any element $K \in \mathcal{T}$, define the following local *a posteriori* error estimator based on the solution of (4.2) by

$$\eta_{m,K} = \|k^{1/2}(\boldsymbol{\rho}_m + k^{-1}\boldsymbol{\sigma}_m)\|_{0,K}.$$

Then the corresponding global *a posteriori* error estimator is

$$\eta_m = \|k^{1/2}(\boldsymbol{\rho}_m + k^{-1}\boldsymbol{\sigma}_m)\|_{0,\Omega} = \min_{\boldsymbol{\tau} \in ND_2} \|k^{1/2}(\boldsymbol{\tau} + k^{-1}\boldsymbol{\sigma}_m)\|_{0,\Omega}. \tag{5.1}$$

This estimator essentially measures the violation of the continuity of the tangential derivatives of the true solution on the edges by the mixed elements.

Based on the explicit approximation in (4.11), we define explicit local *a posteriori* error estimator by

$$\hat{\eta}_{m,K} = \|k^{1/2}(\hat{\boldsymbol{\rho}}_m + k^{-1}\boldsymbol{\sigma}_m)\|_{0,K}, \tag{5.2}$$

for any $K \in \mathcal{T}$ and explicit global *a posteriori* error estimator by

$$\hat{\eta}_m = \|k^{1/2}(\hat{\boldsymbol{\rho}}_m + k^{-1}\boldsymbol{\sigma}_m)\|_{0,\Omega}. \tag{5.3}$$

It is obvious that

$$\eta_m \le \hat{\eta}_m. \tag{5.4}$$

5.2 Error estimators for nonconforming elements

For any element $K \in \mathcal{T}$, define the following local *a posteriori* error estimator based on the solutions of (4.5) and (4.4) by

$$\eta^2_{nc,K} = c^2\eta^2_{nc,1,K} + (1 - c^2)\eta^2_{nc,2,K} \quad \text{for} \quad 0 < c < 1 \tag{5.5}$$

with $\eta_{nc,1,K}$ and $\eta_{nc,2,K}$ given by

$$\eta_{nc,1,K} = \|k^{-1/2}\boldsymbol{\sigma}_{nc} + k^{1/2}\nabla u_{nc}\|_{0,K} \quad \text{and} \quad \eta_{nc,2,K} = \|k^{1/2}(\boldsymbol{\rho}_{nc} - \nabla u_{nc})\|_{0,K}.$$

Then the corresponding global *a posteriori* error estimator is

$$\eta^2_{nc} = c^2\eta^2_{nc,1} + (1 - c^2)\eta^2_{nc,2} \quad \text{for} \quad 0 < c < 1. \tag{5.6}$$

with

$$\eta_{nc,1} = \|k^{-1/2}\boldsymbol{\sigma}_{nc} + k^{1/2}\nabla_h u_{nc}\|_{0,\Omega} = \min_{\boldsymbol{\tau}\in\mathcal{V}_N} \|k^{-1/2}\boldsymbol{\tau} + k^{1/2}\nabla_h u_{nc}\|_{0,\Omega}, \tag{5.7}$$

$$\eta_{nc,2} = \|k^{1/2}(\boldsymbol{\rho}_{nc} + \nabla_h u_{nc})\|_{0,\Omega} = \min_{\boldsymbol{\tau}\in\mathcal{W}_D} \|k^{1/2}(\boldsymbol{\tau} - \nabla_h u_{nc})\|_{0,\Omega} \tag{5.8}$$

The reason we put a c here is to make the error estimator is comparable with the energy norm of the true error, so the efficiency constant can be close to 1. In the real computation, we can choose $c^2 = 1/2$. This estimator essentially measures the violations of the continuity of both the normal components of the flux and the tangential derivatives of the true solution on the edges by the nonconforming elements.

Next, based on the explicit approximations in (4.13) and (4.16), we define explicit local *a posteriori* error estimator by

$$\hat{\eta}_{nc,K}^2 = c^2\hat{\eta}_{nc,1,K}^2 + (1-c^2)\hat{\eta}_{nc,2,K}^2 \quad \text{for} \quad 0 < c < 1 \tag{5.9}$$

with

$$\hat{\eta}_{nc,1,K} = \|k^{-1/2}\hat{\boldsymbol{\sigma}}_{nc,RT_0} + k^{1/2}\nabla u_{nc}\|_{0,K} \quad \text{and} \quad \hat{\eta}_{nc,2,K} = \|k^{1/2}(\hat{\boldsymbol{\rho}}_{nc,N} - \nabla u_{nc})\|_{0,K}$$

for any $K \in \mathcal{T}$. The corresponding explicit global *a posteriori* error estimator is given by

$$\hat{\eta}_{nc}^2 = c^2\hat{\eta}_{nc,1}^2 + (1-c^2)\hat{\eta}_{nc,2}^2 \quad \text{for} \quad 0 < c < 1 \tag{5.10}$$

with

$$\hat{\eta}_{nc,1} = \|k^{-1/2}\hat{\boldsymbol{\sigma}}_{nc,RT_0} + k^{1/2}\nabla u_{nc}\|_{0,\Omega} \quad \text{and} \quad \hat{\eta}_{nc,2} = \|k^{1/2}(\hat{\boldsymbol{\rho}}_{nc,N} - \nabla u_{nc})\|_{0,\Omega}.$$

Let u_{nc} and $\tilde{u}_{nc}$ be the solutions of (3.6) with the right-hand sides f and f_h, respectively, where f_h is piecewise constant with $f_h|_K = \dfrac{1}{|K|}\displaystyle\int_K f\,dx$ for all $K \in \mathcal{T}$. It is easy to see that

$$\|k^{1/2}\nabla_h(u_{nc} - \tilde{u}_{nc})\|_{0,\Omega} \le C\,\|k^{-1/2}h(f - f_h)\|_{0,\Omega}. \tag{5.11}$$

Let $\boldsymbol{\sigma}_m \in RT_0$ be the mixed finite element approximation and let $\mathbf{x}_K$ be the center of inertia of K. By the well-known fact [32] that $(\boldsymbol{\sigma}_m + k\nabla_h\tilde{u}_{nc})|_K = -\dfrac{1}{2}\,f_h|_K(\mathbf{x} - \mathbf{x}_K)$, we have

$$(\boldsymbol{\sigma}_m + k\nabla_h u_{nc})|_K = -\frac{1}{2}\,f_h|_K(\mathbf{x} - \mathbf{x}_K) + (k\nabla_h u_{nc} - k\nabla_h\tilde{u}_{nc})_K.$$

Hence, to avoid the flux recovery, we may replace $\eta_{nc,1,K}$ by

$$\eta_{nc,f,K} = \frac{1}{2}\,\|k^{-1/2}f_h(\mathbf{x} - \mathbf{x}_K)\|_{0,K}$$

to obtain the following estimators

$$\tilde{\eta}_{nc,K}^2 = \eta_{nc,f,K}^2 + \eta_{nc,2,K}^2 \ \forall\, K \in \mathcal{T}, \qquad \tilde{\eta}_{nc}^2 = c^2\eta_{nc,f}^2 + (1-c^2)\eta_{nc,2}^2 \tag{5.12}$$

and

$$\bar{\eta}_{nc,K}^2 = \eta_{nc,f,K}^2 + \hat{\eta}_{nc,2,K}^2 \ \forall\, K \in \mathcal{T}, \qquad \bar{\eta}_{nc}^2 = c^2 \eta_{nc,f}^2 + (1-c^2)\hat{\eta}_{nc,2}^2, \tag{5.13}$$

where $\eta_{nc,f} = \dfrac{1}{2}\left(\sum_{K\in\mathcal{T}} \|k^{-1/2} f_h(\mathbf{x}-\mathbf{x}_K)\|_{0,K}^2\right)^{1/2}$. Now, it follows from (5.7), (5.11), and the triangle inequality that

$$\eta_{nc} \le \hat{\eta}_{nc}, \quad \eta_{nc} \le \tilde{\eta}_{nc} + C\,\|k^{-1/2}h(f-f_h)\|_{0,\Omega}, \text{ and } \eta_{nc} \le \bar{\eta}_{nc} + C\,\|k^{-1/2}h(f-f_h)\|_{0,\Omega}. \tag{5.14}$$

6 Clément-type Interpolations

Clément-type interpolation operators (see, e.g., [8, 38]) are often used for establishing the reliability bound of a posteriori error estimators. We define a weighted Clément-type interpolation operator and to state its approximation and stability properties (see more details in [13]).

Denote by $\mathcal{N}$ and $\mathcal{N}_K$ the sets of all vertices of the triangulation $\mathcal{T}$ and of element $K \in \mathcal{T}$, respectively. For any $z \in \mathcal{N}$, denote by ϕ_z the nodal basis function, let $\omega_z = \text{suppt}\,(\phi_z)$, and denote by $\hat{\omega}_z$ the union of elements in ω_z where the coefficient k_K achieves the maximum for $K \subset \omega_z$. For a given function v, define its weighted average over $\hat{\omega}_z$ by

$$⨍_{\hat{\omega}_z} v\,dx = \frac{\int_{\hat{\omega}_z} v\,\phi_z\,dx}{\int_{\hat{\omega}_z} \phi_z\,dx}. \tag{6.1}$$

Now, following [8], define the interpolation operator $\mathcal{J} : L^2(\Omega) \to \mathcal{U}_D$ by

$$\mathcal{J}v = \sum_{z\in\mathcal{N}} (\pi_z v)\phi_z(x), \tag{6.2}$$

where the nodal value at z is defined by

$$(\mathcal{J}v)(z) = \pi_z v = \begin{cases} ⨍_{\hat{\omega}_z} v\,dx & z \in \mathcal{N}\setminus\Gamma_D, \\ 0 & z \in \mathcal{N}\cap\Gamma_D. \end{cases}$$

In this and next sections, assume that the Hypothesis 2.7 in [8] holds. That is, assume that any two different subdomains $\bar{\Omega}_i$ and $\bar{\Omega}_j$, which share at least one point, there is a connected path passing from $\bar{\Omega}_i$ to $\bar{\Omega}_j$ through adjacent subdomains such that the diffusion coefficient $k(x)$ is monotone along this path. This assumption is weakened to the quasi-monotonicity in [38].

Lemma 6.1. [13] *For any $K \in \mathcal{T}$ and $v \in H_D^1(\Omega)$, the following estimates hold*

$$\|v - \mathcal{J}v\|_{0,K} \le C\,h_K k_K^{-1/2} \|k^{1/2}\nabla v\|_{0,\Delta_K} \tag{6.3}$$

and

$$\|\nabla(v - \mathcal{J}v)\|_{0,K} \le C\,k_K^{-1/2} \|k^{1/2}\nabla v\|_{0,\Delta_K}, \tag{6.4}$$

where Δ_K is the union of all elements that share at least one vertex with K.

Lemma 6.2. [13] *For any $v \in H^1_D(\Omega)$, there exists a positive constant C such that*

$$|(f,\, v - \mathcal{J}v)| \le C\, H_f\, \|k^{1/2}\nabla v\|_{0,\Omega}. \tag{6.5}$$

with

$$H_f = \left(\sum_{z\in\mathcal{N}\cap(F\cup\Gamma_D)} \sum_{K\subset\omega_z} k_K^{-1} h_K^2\, \|f\|_{0,K}^2 + \sum_{z\in\mathcal{N}\setminus(F\cup\Gamma_D)} \sum_{K\subset\omega_z} k_K^{-1} h_K^2\, \|f - \fint_{\omega_z} f\, dx\|_{0,K}^2 \right)^{1/2}.$$

Remark 6.3. *The second term in H_f is a higher order term for $f \in L^2(\Omega)$ and so is the first term for $f \in L^p(\Omega)$ with $p > 2$ (see [18]).*

Denote by $\tilde{\omega}_z$ the union of elements in ω_z where the coefficient k_K achieves the minimum for $K \subset \omega_z$. Similarly, we can define a robust interpolation $\mathcal{J}' : L^2(\Omega) \to \mathcal{U}_N$, where $\mathcal{U}_N = \{v \in \mathcal{U} : v = 0 \text{ on } \Gamma_N\}$. Similar proofs as those of Lemmas 6.1 and 6.2 (see [13]) yield the following properties.

Lemma 6.4. *For any $K \in \mathcal{T}$ and $v \in H^1(\Omega)$, the following estimates hold*

$$\|v - \mathcal{J}'v\|_{0,K} \le C\, h_K k_K^{1/2} \|k^{-1/2}\nabla^\perp v\|_{0,\Delta_K} \tag{6.6}$$

and

$$\|\nabla^\perp(v - \mathcal{J}'v)\|_{0,K} \le C\, k_K^{1/2} \|k^{-1/2}\nabla^\perp v\|_{0,\Delta_K}. \tag{6.7}$$

Lemma 6.5. *For any $v \in H^1(\Omega)$, there exists a positive constant C such that*

$$|(f,\, v - \mathcal{J}'v)| \le C\, G_f\, \|k^{-1/2}\nabla^\perp v\|_{0,\Omega} \tag{6.8}$$

where

$$G_f = \left(\sum_{z\in\mathcal{N}\cap(F\cup\Gamma_N)} \sum_{K\subset\omega_z} k_K h_K^2\, \|f\|_{0,K}^2 + \sum_{z\in\mathcal{N}\setminus(F\cup\Gamma_N)} \sum_{K\subset\omega_z} k_K h_K^2\, \|f - \fint_{\omega_z} f\, dx\|_{0,K}^2 \right)^{1/2}.$$

7 Reliability and Efficiency Bounds

This section analyzes the estimators introduced in section 5 by establishing the reliability and efficiency bounds with constants independent of the size of jumps. It is now a standard technique (see, e.g., [5]) to analyze estimators for the mixed and nonconforming elements by using the Helmholtz decomposition (see, e.g., [26]) stated in the following lemma.

Lemma 7.1. *For a vector-valued function $\boldsymbol{\tau} \in L^2(\Omega)^2$, there exist $\alpha \in H^1_D(\Omega)$ and $\beta \in H^1_N(\Omega)$ such that*

$$\boldsymbol{\tau} = k(x)\nabla\alpha + \nabla^\perp\beta. \tag{7.1}$$

Integrating by parts gives

$$(\nabla\alpha,\, \nabla^\perp\beta) = 0 \tag{7.2}$$

for all $\alpha \in H^1_D(\Omega)$ and all $\beta \in H^1_N(\Omega)$. Hence, we have

$$(k^{-1}\boldsymbol{\tau}, \boldsymbol{\tau}) = (k\nabla\alpha, \nabla\alpha) + (k^{-1}\nabla^\perp\beta, \nabla^\perp\beta). \tag{7.3}$$

7.1 Reliability on mixed elements

Theorem 7.2. *The estimator η_m defined in* (5.1) *satisfies the following global reliability bound:*

$$\|k^{-1/2}E_m\|_{0,\Omega} \le C\left(\eta_m + \|k^{-1/2}h(f-f_h)\|_{0,\Omega} + G_{\nabla_h\times(k^{-1}\boldsymbol{\sigma}_m)}\right). \tag{7.4}$$

If $\nabla_h \times (k^{-1}\boldsymbol{\sigma}_m) = 0$, then

$$\|k^{-1/2}E_m\|_{0,\Omega} \le C\left(\eta_m + \|k^{-1/2}h(f-f_h)\|_{0,\Omega}\right). \tag{7.5}$$

Proof. Let $E_m = \boldsymbol{\sigma} - \boldsymbol{\sigma}_m = -k\nabla u - \boldsymbol{\sigma}_m \in H_N(\mathrm{div};\Omega)$, by Lemma 7.1 and (7.3), there exist $\alpha_m \in H^1_D(\Omega)$ and $\beta_m \in H^1_N(\Omega)$ such that

$$E_m = k\nabla\alpha_m + \nabla^\perp\beta_m$$

and that

$$\|k^{-1/2}E_m\|^2_{0,\Omega} = \|k^{1/2}\nabla\alpha_m\|^2_{0,\Omega} + \|k^{-1/2}\nabla^\perp\beta_m\|^2_{0,\Omega}. \tag{7.6}$$

The upper bound of the first term in (7.6) follows easily from (7.2), integration by parts, $E_m\cdot\mathbf{n} = 0$ on Γ_N, $\alpha_m = 0$ on Γ_D, the first equation in (3.3), and the Cauchy-Schwarz inequality that

$$\begin{aligned}\|k^{1/2}\nabla\alpha_m\|^2_{0,\Omega} &= (E_m,\,\nabla\alpha_m) = -(\nabla\cdot E_m,\,\alpha_m) = (\nabla\cdot E_m,\,Q_h\alpha_m - \alpha_m)\\ &= (f-f_h,\,Q_h\alpha_m-\alpha_m) \le C\,\|k^{-1/2}h(f-f_h)\|_{0,\Omega}\|k^{1/2}\nabla\alpha_m\|_{0,\Omega},\end{aligned}$$

which implies

$$\|k^{1/2}\nabla\alpha_m\|^2_{0,\Omega} \le C\,\|k^{-1/2}h(f-f_h)\|^2_{0,\Omega}. \tag{7.7}$$

To bound the second term in (7.6), $\|k^{-1/2}\nabla^\perp\beta_m\|^2_{0,\Omega}$, notice first that $\nabla^\perp\mathcal{U}_N \subset RT_0$. Then the first equation in (3.3) gives

$$(k^{-1}E_m,\,\nabla^\perp v) = (\nabla\cdot\nabla^\perp v,\,e_m) = 0 \quad \forall\, v\in\mathcal{U}_N. \tag{7.8}$$

By (7.2), the fact that $\mathcal{J}'\beta_m \in \mathcal{U}_N$, (7.8), (1.3), boundary conditions of $\boldsymbol{\rho}_m - \nabla u$ on Γ_D and $(I-\mathcal{J}')\beta_m$ on Γ_N, the Cauchy-Schwarz inequality, and (6.7), we have

$$\begin{aligned}&\|k^{-1/2}\nabla^\perp\beta_m\|^2_{0,\Omega} = (k^{-1}E_m,\,\nabla^\perp\beta_m) = (k^{-1}E_m,\,\nabla^\perp(\beta_m - \mathcal{J}'\beta_m))\\ =\;& \left(\boldsymbol{\rho}_m - \nabla u,\,\nabla^\perp(I-\mathcal{J}')\beta_m\right) - \left(\boldsymbol{\rho}_m + k^{-1}\boldsymbol{\sigma}_m,\,\nabla^\perp(I-\mathcal{J}')\beta_m\right)\\ =\;& -\left(\nabla\times\boldsymbol{\rho}_m,\,(I-\mathcal{J}')\beta_m\right) - \left(\boldsymbol{\rho}_m + k^{-1}\boldsymbol{\sigma}_m,\,\nabla^\perp(I-\mathcal{J}')\beta_m\right)\\ \le\;& -\left(\nabla\times\boldsymbol{\rho}_m,\,(I-\mathcal{J}')\beta_m\right) + C\,\eta_m\,\|k^{-1/2}\nabla^\perp\beta_m\|_{0,\Omega}\\ =\;& -\left(\nabla_h\times(\boldsymbol{\rho}_m + k^{-1}\boldsymbol{\sigma}_m),\,(I-\mathcal{J}')\beta_m\right) + \left(\nabla_h\times(k^{-1}\boldsymbol{\sigma}_m),\,(I-\mathcal{J}')\beta_m\right)\\ &+C\,\eta_m\,\|k^{-1/2}\nabla^\perp\beta_m\|_{0,\Omega}.\end{aligned}$$

Using the Cauchy-Schwartz inequality, the inverse inequality, (6.6), and (6.8), we obtain that

$$\begin{aligned}
&\big(\nabla_h \times (\boldsymbol{\rho}_m + k^{-1}\boldsymbol{\sigma}_m),\, (I - \mathcal{J}')\beta_m\big) \\
\le\;& C\,\|k^{1/2} h \nabla_h \times (\boldsymbol{\rho}_m + k^{-1}\boldsymbol{\sigma}_m)\|_{0,\Omega}\,\|k^{-1/2} h^{-1}(I - \mathcal{J}')\beta_m\|_{0,\Omega} \\
\le\;& C\,\eta_m\,\|k^{-1/2}\nabla^{\perp}\beta_m\|_{0,\Omega}
\end{aligned}$$

and that

$$\big(\nabla_h \times (k^{-1}\boldsymbol{\sigma}_m), (I - \mathcal{J}')\beta_m\big) \le C\, G_{\nabla_h \times (k^{-1}\boldsymbol{\sigma}_m)} \|k^{-1/2}\nabla^{\perp}\beta_m\|_{0,\Omega_z}.$$

Combining the above three inequalities and dividing the quantity $\|k^{-1/2}\nabla^{\perp}\beta_m\|_{0,\Omega}$ and squaring on both sides give

$$\|k^{-1/2}\nabla^{\perp}\beta_m\|_{0,\Omega}^2 \le C\left(\eta_m + G_{\nabla_h \times (k^{-1}\boldsymbol{\sigma}_m)}\right)^2,$$

which, together with (7.6) and (7.7), yields (7.4).

Finally, (7.5) is a direct consequence of (7.4) and the fact that $G_0 = 0$. This completes the proof of the theorem. □

Remark 7.3. *In the case that $\mathcal{V}_N = RT_0$, the condition*

$$\nabla_h \times (k^{-1}\boldsymbol{\sigma}_m) = 0$$

holds.

Corollary 7.4. *The reliability bounds in* Theorem 7.2 *hold for the explicit error estimator $\hat{\eta}_m$ as well.*

Proof. The corollary is a direct consequence of Theorem 7.2 and (5.4). □

7.2 Reliability on nonconforming elements

Theorem 7.5. *The estimator η_{nc} defined in* (5.6) *satisfies the following global reliability bound:*

$$\|k^{1/2}\nabla_h e_{nc}\|_{0,\Omega} \le C\,(\eta_{nc} + H_f) \tag{7.9}$$

Proof. Let $e_{nc} = u - u_{nc}$, by Lemma 7.1 and (7.3), there exist $\alpha_{nc} \in H_D^1(\Omega)$ and $\beta_{nc} \in H_N^1(\Omega)$ such that

$$k\nabla_h e_{nc} = k\nabla\alpha_{nc} + \nabla^{\perp}\beta_{nc}$$

and that

$$\|k^{1/2}\nabla_h e\|_{0,\Omega}^2 = \|k^{1/2}\nabla\alpha_{nc}\|_{0,\Omega}^2 + \|k^{-1/2}\nabla^{\perp}\beta_{nc}\|_{0,\Omega}^2 \tag{7.10}$$

By using (3.8), integrations by parts, boundary conditions on $k\nabla u + \boldsymbol{\sigma}_{nc}$ and $(I - \mathcal{J})\alpha_{nc}$, the Cauchy-Schwarz inequality, (6.4), the fact that $\nabla_h \cdot (k\nabla_h u_{nc}) = 0$, the inverse inequality, (6.3), and (6.5), we have

$$
\begin{aligned}
&\|k^{1/2}\nabla\alpha_{nc}\|_{0,\Omega}^2 = (k\nabla_h e_{nc},\, \nabla\alpha_{nc}) = (k\nabla_h e_{nc},\, \nabla(I-\mathcal{J})\alpha_{nc})) \\
= \;& (k\nabla u + \boldsymbol{\sigma}_{nc},\, \nabla(I-\mathcal{J})\alpha_{nc})) - (\boldsymbol{\sigma}_{nc} + k\nabla_h u_{nc},\, \nabla(I-\mathcal{J})\alpha_{nc}) \\
\le \;& (f - \nabla\cdot\boldsymbol{\sigma}_{nc},\, \alpha_{nc} - \mathcal{J}\alpha_{nc}) + C\,\eta_{nc,1}\,\|k^{1/2}\nabla\alpha_{nc}\|_{0,\Omega} \\
= \;& (f,\, \alpha_{nc} - \mathcal{J}\alpha_{nc}) - (\nabla_h\cdot(\boldsymbol{\sigma}_{nc} + k\nabla_h u_{nc}),\, \alpha_{nc} - \mathcal{J}\alpha_{nc}) + C\,\eta_{nc,1}\,\|k^{1/2}\nabla\alpha_{nc}\|_{0,\Omega} \\
\le \;& C\,(\eta_{nc,1} + H_f)\,\|k^{1/2}\nabla\alpha_{nc}\|_{0,\Omega}.
\end{aligned}
$$

To bound the second term in (7.10), first by integration by parts and the orthogonality of nonconforming elements, i.e., $\int_e [u_{nc}]\,ds = 0$ for any $e \in \mathcal{E}$, we have

$$
\begin{aligned}
(\nabla_h u_{nc},\, \nabla^\perp\mathcal{J}'\beta_{nc}) &= \sum_{K\in\mathcal{T}} \int_{e\in\partial K} u_{nc}\,(\nabla^\perp\mathcal{J}'\beta_{nc})\cdot\mathbf{n}\,ds \\
&= \sum_{e\in\mathcal{E}} (\nabla^\perp\mathcal{J}'\beta_{nc})\cdot\mathbf{n} \int_e [u_{nc}]\,ds = 0.
\end{aligned}
$$

It then follows from integration by parts, boundary conditions on $\boldsymbol{\rho}_{nc}$ and $(I-\mathcal{J}')\beta_{nc}$, the fact that $\nabla_h \times (k^{-1}\nabla_h u_{nc}) = 0$, the Cauchy-Schwarz and inverse inequalities, (6.7), and (6.6) that

$$
\begin{aligned}
&\|k^{-1/2}\nabla^\perp\beta_{nc}\|_{0,\Omega}^2 = (\nabla_h e,\, \nabla^\perp\beta_{nc}) = (-\nabla_h u_{nc},\, \nabla^\perp(I-\mathcal{J}')\beta_{nc}) \\
= \;& (\boldsymbol{\rho}_{nc} - \nabla_h u_{nc},\, \nabla^\perp(I-\mathcal{J}')\beta_{nc}) - (\boldsymbol{\rho}_{nc},\, \nabla^\perp(I-\mathcal{J}')\beta_{nc}) \\
= \;& (\boldsymbol{\rho}_{nc} - \nabla_h u_{nc},\, \nabla^\perp(I-\mathcal{J}')\beta_{nc}) + \big(\nabla_n \times (\boldsymbol{\rho}_{nc} - \nabla_h u_{nc}),\, (I-\mathcal{J}')\beta_{nc}\big) \\
\le \;& C\,\eta_{nc,2}\,\|k^{-1/2}\nabla^\perp\beta_{nc}\|_{0,\Omega}.
\end{aligned}
$$

Now, the reliability bound in (7.9) is a direct consequence of the above two inequalities. This completes the proof of the theorem. □

Corollary 7.6. *Let η denote the estimators $\tilde{\eta}_{nc}$, $\hat{\eta}_{nc}$, or $\bar{\eta}_{nc}$, then we have the following reliability bound*

$$
\|k^{1/2}\nabla_h e_{nc}\|_{0,\Omega} \le C\left(\eta + H_f + \|k^{-1/2}h(f - f_h)\|_{0,\Omega}\right).
$$

Proof. The corollary is a direct consequence of Theorem 7.5 and (5.14). □

7.3 Efficiency

To establish the efficiency bounds, we make use of the known result on the edge error estimators in [16]. To this end, for any $e \in \mathcal{E}_\Omega$ and any vector-valued function $\boldsymbol{\tau}$ that is piecewise linear with respect to the triangulation $\mathcal{T}$, denote the jump of the normal and tangential components of $\boldsymbol{\tau}$ across $e = K_e^+ \cap K_e^-$ by

$$J_{\mathbf{n},e}(\boldsymbol{\tau}) = [\boldsymbol{\tau} \cdot \mathbf{n}_e] = (\boldsymbol{\tau}|_{K_e^+} - \boldsymbol{\tau}|_{K_e^-}) \cdot \mathbf{n}_e \quad \text{and} \quad J_{\mathbf{t},e}(\boldsymbol{\tau}) = [\boldsymbol{\tau} \cdot \mathbf{t}_e] = (\boldsymbol{\tau}|_{K_e^+} - \boldsymbol{\tau}|_{K_e^-}) \cdot \mathbf{t}_e,$$

respectively. For any $e \in \mathcal{E} \backslash \mathcal{E}_\Omega$, we set

$$J_{\mathbf{n},e}(\boldsymbol{\tau}) = 0, \quad \text{and} \quad J_{\mathbf{t},e}(\boldsymbol{\tau}) = 0.$$

7.3.1 Efficiency on mixed elements

We define an edge error estimator for mixed elements as follows:

$$\eta_{m,E} := \left(\sum_{e\in\mathcal{E}} \eta_{m,e}^2 \right)^{1/2} \quad \text{with} \quad \eta_{m,e} = \left(\frac{k_{K_e^+} + k_{K_e^-}}{2} \, h_e \int_e |J_{\mathbf{t},e}(k^{-1}\boldsymbol{\sigma}_m)|^2 \, ds \right)^{1/2}. \tag{7.11}$$

Proposition 7.7. *There exists a constant $C > 0$ such that*

$$\eta_{m,e} \le C \, \|k^{-1/2} \nabla^\perp \beta_m\|_{0,\omega_e} \tag{7.12}$$

and that

$$\eta_{m,E} \le C \, \|k^{-1/2} \nabla^\perp \beta_m\|_{0,\Omega} \le C \, \|k^{-1/2} E_m\|_{0,\Omega}. \tag{7.13}$$

Proof. A similar proof as those of Lemmas 6.1 and 6.2 in [15] shows that there exists a constant $C > 0$ impendent of k such that

$$h_e \int_e |J_{\mathbf{t},e}(k^{-1}\boldsymbol{\sigma}_m)|^2 \, ds \le C \, \|k^{-1} \nabla^\perp \beta_m\|_{0,\omega_e}^2.$$

Hence,

$$\begin{aligned} \eta_{m,e}^2 &\le C \, (k_{K_e^+} + k_{K_e^-}) \|k^{-1} \nabla^\perp \beta_m\|_{0,\omega_e}^2 \\ &= C \, (k_{K_e^+} + k_{K_e^-})(k_{K_e^+}^{-1} \, \|k^{-1/2} \nabla^\perp \beta_m\|_{0,K_e^+}^2 + k_{K_e^-}^{-1} \, \|k^{-1/2} \nabla^\perp \beta_m\|_{0,K_e^-}^2) \\ &\le C \, \|k^{-1/2} \nabla^\perp \beta_m\|_{0,\omega_e}^2, \end{aligned}$$

which proves (7.12). Summing it over all edges $e \in \mathcal{E}$ leads to (7.13). □

Theorem 7.8. *The following local efficiency bound for the explicit error estimator $\hat{\eta}_{m,K}$ holds:*

$$\hat{\eta}_{m,K}^2 \le C \, \|k^{-1/2} \nabla^\perp \beta_m\|_{0,\omega_K}, \tag{7.14}$$

where ω_K is the union of elements sharing a common edge with K. The following global efficiency bound holds for all error estimators:

$$\eta_m \le \hat{\eta}_m \le C \, \|k^{-1/2} \nabla^\perp \beta_m\|_{0,\Omega} \le C \, \|k^{1/2} E_m\|_{0,\Omega}. \tag{7.15}$$

Proof. (7.15) follows straightforward from (5.4) and (7.14). To show the validity of (7.14), by Proposition 7.7 it suffices to prove that for any element $K \in \mathcal{T}$

$$\hat{\eta}_{m,K}^2 \le C \sum_{e\in\partial K} \eta_{m,e}^2 = C \sum_{e\in\partial K\setminus\partial\Omega} \frac{h_e}{2} (k_{K_e^+} + k_{K_e^-}) \int_e |J_{t,e}(k^{-1}\boldsymbol{\sigma}_m)|^2\, ds. \tag{7.16}$$

To do so, for any edge $e \in \partial K$, without loss of generality let K be K_e^+ and let K_e^- be the adjacent element with the common edge e. Since $\boldsymbol{\tau} = k^{-1}\boldsymbol{\sigma}_m$ is piecewise linear, we have

$$\boldsymbol{\tau}|_K = \sum_{e\in\partial K} \left(\tau_{e,1,K}\, \boldsymbol{\psi}_{e,1} + \tau_{e,2,K}\, \boldsymbol{\psi}_{e,2}\right),$$

where $\tau_{e,i,K} = \int_e s^{i-1}(\boldsymbol{\tau}\cdot\mathbf{t}_e)_K\, ds$ is the $(i-1)$-th moment of the tangential component of $\boldsymbol{\tau}$ on e and $\boldsymbol{\psi}_{e,i}$ is the nodal basis function of ND_2. For any $\mathbf{x} \in K$,

$$\begin{aligned}
\hat{\boldsymbol{\rho}}_m + \boldsymbol{\tau} &= \sum_{e\in\partial K} (\hat{\rho}_{e,1} - \tau_{e,1,K})\ \boldsymbol{\psi}_{e,1} + (\hat{\rho}_{e,2} - \tau_{e,2,K})\ \boldsymbol{\psi}_{e,2} \\
&= \sum_{e\in\partial K\setminus\partial\Omega} (1-\gamma_{1,e}) \left(\left(\tau_{e,1,K_e^-} - \tau_{e,1,K}\right) \boldsymbol{\psi}_{e,1} + \left(\tau_{e,2,K_e^-} - \tau_{e,2,K}\right) \boldsymbol{\psi}_{e,2}\right) \\
&= \sum_{e\in\partial K\setminus\partial\Omega} (\gamma_{1,e} - 1) \left(\boldsymbol{\psi}_{e,1} \int_e J_{\mathbf{t},e}(\boldsymbol{\tau}) ds + \boldsymbol{\psi}_{e,2} \int_e s J_{\mathbf{t},e}(\boldsymbol{\tau}) ds\right) \\
&= \sum_{e\in\partial K\setminus\partial\Omega} (\gamma_{1,e} - 1) \left(j_{e,1}\, \boldsymbol{\psi}_{e,1} + j_{e,2}\, \boldsymbol{\psi}_{e,2}\right)
\end{aligned}$$

with $j_{e,i} = \int_e s^{i-1} J_{\mathbf{t},e}(\boldsymbol{\tau}) ds$. Since $J_{\mathbf{t},e}(\boldsymbol{\tau})$ is a linear function on e, Lemma 4.2 gives

$$J_{\mathbf{t},e}(\boldsymbol{\tau}) = j_{e,1}\,(\boldsymbol{\psi}_{e,1}\cdot\mathbf{t}_e) + j_{e,2}\,(\boldsymbol{\psi}_{e,2}\cdot\mathbf{t}_e)$$

and

$$\int_e |J_{\mathbf{t},e}(\boldsymbol{\tau})|^2 ds = j_{e,1}^2 \int_e |\boldsymbol{\psi}_{e,1}|^2 ds + j_{e,2}^2 \int_e |\boldsymbol{\psi}_{e,2}|^2 ds.$$

Now, by the triangle inequality and the fact that $\int_K |\boldsymbol{\psi}_{e,i}|^2 dx \le C h_e \int_e |\boldsymbol{\psi}_{e,i}|^2 ds$, we have

$$\begin{aligned}
\hat{\eta}_{m,K} &= \|k^{1/2}(\hat{\boldsymbol{\rho}}_m + \boldsymbol{\tau})\|_{0,K} \\
&\le C \left(k_K \sum_{e\in\partial K\setminus\partial\Omega} (1-\gamma_{1,e}) \left(j_{e,1}^2 \int_K |\boldsymbol{\psi}_{e,1}|^2\, dx + j_{e,2}^2 \int_K |\boldsymbol{\psi}_{e,2}|^2\, dx\right)\right)^{1/2} \\
&\le C \left(k_K \sum_{e\in\partial K\setminus\partial\Omega} h_e \int_e |J_{t,e}(\boldsymbol{\tau})|^2\, ds\right)^{1/2} \le C \sum_{e\in\partial K\setminus\partial\Omega} \eta_{m,e}^2.
\end{aligned}$$

This proves (7.16) and, hence, the theorem. □

7.3.2 Efficiency on nonconforming elements

A weighted edge error estimator for nonconforming elements is defined by

$$\eta_{nc,E} := \left(\sum_{e\in\mathcal{E}} \eta_{nc,e}^2 \right)^{1/2}$$

with

$$\eta_{nc,e}^2 = \frac{2h_e}{k_{K_e^+} + k_{K_e^-}} \int_e |J_{\mathbf{n},e}(k\nabla_h u_{nc})|^2 \, ds + \frac{h_e \, k_{K_e^+} \, k_{K_e^-}}{k_{K_e^+} + k_{K_e^-}} \int_e |J_{\mathbf{t},e}(\nabla_h u_{nc})|^2 \, ds. \tag{7.17}$$

Theorem 7.9. *There exists a constant $C > 0$ such that*

$$\eta_{nc,e}^2 \le C \left(\|k^{1/2}\nabla_h e_{nc}\|_{0,\omega_e}^2 + \sum_{K\in\mathcal{T}\cap\omega_e} \frac{h_K^2}{k_K} \|f - f_h\|_{0,K}^2 \right) \tag{7.18}$$

and

$$\eta_{nc,E} \le C \, \|k^{1/2}\nabla_h e_{nc}\|_{0,\Omega} + C \left(\sum_{K\in\mathcal{T}} \frac{h_K^2}{k_K} \|f - f_h\|_{0,K}^2 \right)^{1/2}. \tag{7.19}$$

Proof. A similar proof as that for the conforming linear elements in [38] shows that the first term of $\eta_{nc,e}^2$ defined in (7.17) satisfies (7.18). It then suffices to show that (7.18) holds for the second term of $\eta_{nc,e}^2$. To do so, we recall the following inequality (estimate (3.3) in [17]):

$$h_e \int_e |J_{\mathbf{t},e}(\nabla_h u_{nc})|^2 \, ds \le C \, \|\nabla_h e_{nc}\|_{0,\omega_e}^2, \tag{7.20}$$

which holds with a constant $C > 0$ independent of the jump of k. Hence,

$$\begin{aligned} &\frac{h_e \, k_{K_e^+} \, k_{K_e^-}}{k_{K_e^+} + k_{K_e^-}} \int_e |J_{\mathbf{t},e}(\nabla_h u_{nc})|^2 \, ds \le C \frac{k_{K_e^+} \, k_{K_e^-}}{k_{K_e^+} + k_{K_e^-}} \|\nabla_h e_{nc}\|_{0,\omega_e}^2 \\ \le\ & C \left(\frac{k_{K_e^-}}{k_{K_e^+} + k_{K_e^-}} \|k^{1/2}\nabla_h e\|_{0,K_e^+}^2 + \frac{k_{K_e^+}}{k_{K_e^+} + k_{K_e^-}} \|k^{1/2}\nabla_h e_{nc}\|_{0,K_e^-}^2 \right) \\ \le\ & C \, \|k^{1/2}\nabla_h e_{nc}\|_{0,\omega_e}^2. \end{aligned}$$

This proves (7.18). The global bound (7.19) is a direct result of (7.18). □

Theorem 7.10. *The following local efficiency bound for the explicit error estimator $\hat{\eta}_{nc,K}$ defined in* (5.9) *holds:*

$$\hat{\eta}_{nc,K}^2 \le C \left(\|k^{1/2}\nabla_h e_{nc}\|_{0,\omega_K}^2 + \sum_{T\in\omega_K} \frac{h_T^2}{k_T} \|f - f_h\|_{0,T}^2 \right), \tag{7.21}$$

where ω_K is the union of elements sharing a common edge with K. The following global efficiency bound for both error estimators η_{nc} and $\hat{\eta}_{nc}$ holds:

$$\eta_{nc} \le \hat{\eta}_{nc} \le C \, \|k^{1/2}\nabla_h e_{nc}\|_{0,\Omega} + C \left(\sum_{K\in\mathcal{T}} \frac{h_K^2}{k_K} \|f - f_h\|_{0,K}^2 \right)^{1/2}. \tag{7.22}$$

Proof. (7.22) is a direct consequence of (5.14) and (7.21). To show the validity of (7.21), by Theorem 7.9 it suffices to prove that for any element $K \in \mathcal{T}$

$$\hat{\eta}_{nc,K}^2 \quad \le \quad C \sum_{e\in\partial K} \eta_{nc,e}^2. \tag{7.23}$$

To this end, for any edge $e \in \partial K$, without loss of generality let $\mathbf{n}_e$ be the outward unit vector normal to ∂K and denote by K_e the adjacent element with the common edge e. Let $\boldsymbol{\tau}_2 = -k\nabla_h u_{nc}$ that is piecewise constant, then $\boldsymbol{\tau}_2|_K$ may be represented in terms of, $\{\boldsymbol{\phi}_e\}_{e\in\partial K}$, the nodal basis function of RT_0:

$$\boldsymbol{\tau}_2|_K = \sum_{e\in\partial K} \tau_{2,e,K}\, \boldsymbol{\phi}_e.$$

For any $\mathbf{x} \in K$, (4.13) and (4.14) give

$$\begin{aligned} \hat{\boldsymbol{\sigma}}_{nc} - \boldsymbol{\tau}_2 \quad &= \quad \sum_{e\in\partial K} (\hat{\sigma}_e - \tau_{2,e,K})\ \boldsymbol{\phi}_e = \sum_{e\in\partial K\setminus\partial\Omega} (\gamma_{2,e} - 1)\,(\tau_{2,e,K} - \tau_{2,e,K_e})\ \boldsymbol{\phi}_e \\ &= \quad \sum_{e\in\partial K\setminus\partial\Omega} (\gamma_{2,e} - 1) J_{\mathbf{n},e}(\boldsymbol{\tau}_2)\, \boldsymbol{\phi}_e. \end{aligned}$$

Similarly, for $\boldsymbol{\tau}_3 = -\nabla_h u_{nc}$ and any $\mathbf{x} \in K$, we have

$$\begin{aligned} \hat{\boldsymbol{\rho}}_{nc} - \boldsymbol{\tau}_3 \quad &= \quad \sum_{e\in\partial K} (\hat{\rho}_e - \tau_{3,e,K})\ \boldsymbol{\psi}_e(\mathbf{x}) = \sum_{e\in\partial K\setminus\partial\Omega} (\gamma_{3,e} - 1)\,(\tau_{3,e,K} - \tau_{3,e,K_e})\ \boldsymbol{\psi}_e \\ &= \quad \sum_{e\in\partial K\setminus\partial\Omega} (\gamma_{3,e} - 1) J_{\mathbf{t},e}(\boldsymbol{\tau}_3)\, \boldsymbol{\psi}_e. \end{aligned}$$

Now, it follows from the triangle inequality, the facts that

$$\int_K |\boldsymbol{\phi}_e|^2\, dx \le C\, h_e^2 \quad \text{and} \quad \int_K |\boldsymbol{\psi}_e|^2\, dx \le C\, h_e^2,$$

(4.15), and (4.18) that

$$
\begin{aligned}
\hat{\eta}_{nc,K}^2 &= c^2\|k^{-1/2}\left(\hat{\boldsymbol{\sigma}}_{nc}-\boldsymbol{\tau}_2\right)\|_{0,K}^2+(1-c^2)\|k^{1/2}\left(\hat{\boldsymbol{\rho}}_{nc}-\boldsymbol{\tau}_3\right)\|_{0,K}^2 \\
&\le C\left(k_K^{-1}\|\hat{\boldsymbol{\sigma}}_{nc}-\boldsymbol{\tau}_2\|_{0,K}^2+k_K\|\hat{\boldsymbol{\sigma}}_{nc}-\boldsymbol{\tau}_3\|_{0,K}^2\right) \\
&\le C\left(\sum_{e\in\partial K\setminus\partial\Omega} 2k_K^{-1}(1-\gamma_{2,e})^2h_e^2|J_{\mathbf{n},e}(\boldsymbol{\tau}_2)|^2+\sum_{e\in\partial K\setminus\partial\Omega} k_K(1-\gamma_{3,e})^2h_e^2|J_{\mathbf{t},e}(\boldsymbol{\tau}_3)|^2\right) \\
&= C\left(\sum_{e\in\partial K\setminus\partial\Omega}\frac{h_e}{\left(k_K^{1/2}+k_{K_e}^{1/2}\right)^2}\left(2\int_e|J_{\mathbf{n},e}(\boldsymbol{\tau}_2)|^2\,ds+k_K\,k_{K_e}\int_e|J_{\mathbf{t},e}(\boldsymbol{\tau}_3)|^2\,ds\right)\right) \\
&\le C\left(\sum_{e\in\partial K\setminus\partial\Omega}\frac{h_e}{k_K+k_{K_e}}\left(2\int_e|J_{\mathbf{n},e}(\boldsymbol{\tau}_2)|^2\,ds+k_K\,k_{K_e}\int_e|J_{\mathbf{t},e}(\boldsymbol{\tau}_3)|^2\,ds\right)\right) \\
&= C\left(\sum_{e\in\partial K\setminus\partial\Omega}\eta_{nc,e}^2\right),
\end{aligned}
$$

which proves (7.23) and, hence, (7.21). This completes the proof of the theorem. □

Theorem 7.11. *There exists a positive constant C such that*

$$\bar{\eta}_{nc}\le C\left(\|k^{1/2}\nabla_h e_{nc}\|_{0,\Omega}+\|k^{1/2}h(f-f_h)\|_{0,\Omega}\right). \tag{7.24}$$

Proof. (7.24) is a direct consequence of Theorem 7.10 and the following inequality (Theorem 3.2 in [2])

$$\hat{\eta}_{nc,f,K}\le C\left(\|k^{1/2}\nabla\alpha_{nc}\|_{0,K}+\|k^{-1/2}h(f-f_h)\|_{0,K}\right).$$

□

8 Numerical Experiments

In this section, we report some numerical results for an interface problem with intersecting interfaces used by many authors, e.g., [27, 33, 19], which is considered as a benchmark test problem. For this test problem, we show numerically that the recovery-based a posteriori error estimators introduced in [16] for both mixed and nonconforming elements over-refine regions along the interfaces and, hence, fail to reduce the global error. For the same test problem, numerical results show that the estimators introduced in this paper are accurate and generate meshes with optimal decay of the error with respect to the number of unknowns.

To this end, let $\Omega=(-1,1)^2$ and

$$u(r,\theta)=r^{\beta}\mu(\theta)$$

in the polar coordinates at the origin with

$$\mu(\theta) = \begin{cases} \cos((\pi/2-\sigma)\beta)\cdot\cos((\theta-\pi/2+\rho)\beta) & \text{if } \ 0\le\theta\le\pi/2, \\ \cos(\rho\beta)\cdot\cos((\theta-\pi+\sigma)\beta) & \text{if } \ \pi/2\le\theta\le\pi, \\ \cos(\sigma\beta)\cdot\cos((\theta-\pi-\rho)\beta) & \text{if } \ \pi\le\theta\le 3\pi/2, \\ \cos((\pi/2-\rho)\beta)\cdot\cos((\theta-3\pi/2-\sigma)\beta) & \text{if } \ 3\pi/2\le\theta\le 2\pi, \end{cases}$$

where σ and ρ are numbers. The function $u(r,\theta)$ satisfies the interface equation in (2.1) with $\Gamma_N = \emptyset$, $f = 0$, and

$$k(x) = \begin{cases} R & \text{in } \ (0,1)^2 \cup (-1,0)^2, \\ 1 & \text{in } \ \Omega \setminus ([0,1]^2 \cup [-1,0]^2). \end{cases}$$

The numbers β, R, σ, and ρ satisfy some nonlinear relations (e.g., [33, 19]). For example, when $\beta = 0.1$, then

$$R \approx 161.4476387975881, \quad \rho = \pi/4, \quad \text{and} \quad \sigma \approx -14.92256510455152.$$

Note that when $\beta = 0.1$, this is a difficult problem for computation.

Remark 8.1. *This problem does not satisfy* Hypothesis 2.7 *in* [8] *and the distribution of its coefficients is not quasi-monotone.*

Note that the solution $u(r,\theta)$ is only in $H^{1+\beta-\epsilon}(\Omega)$ for any $\epsilon > 0$ and, hence, it is very singular for small β at the origin. This suggests that refinement is centered around the origin. In this example we choose $\tilde{\eta}_{nc}$ and $\overline{\eta}_{nc}$ with constant $c^2 = 0.5$ as the implicit and explicit error estimators for the nonconforming method, respectively.

Starting with a coarse triangulation $\mathcal{T}_0$, a sequence of meshes is generated by using standard adaptive meshing algorithm that adopts the Dörfler's bulk marking strategy [24]: construct a *minimal* subset $\hat{\mathcal{T}}$ of $\mathcal{T}$ such that

$$\sum_{K\in\hat{\mathcal{T}}} \eta_K^2 \ge \theta_E^2 \sum_{K\in\mathcal{T}} \eta_K^2 \tag{8.1}$$

with $\theta_E = 0.2$. The choice of $\theta_E = 0.2$ is not critical but recommended in [19] for better performance. Marked triangles are refined regularly by dividing each into four congruent triangles. Additionally, irregularly refined triangles are needed in order to make the triangulation admissible. For more details on adaptive mesh refinement algorithms, see, e.g., [10].

The true errors can be computed by the following identities

$$\text{err}_\text{m} = \|k^{1/2}\nabla u + k^{-1/2}\boldsymbol{\sigma}_m\|_{0,\Omega}^2 = \int_{\partial\Omega} u\,[\mathbf{n}\cdot(k\nabla u + 2\boldsymbol{\sigma}_m)]\,ds + \|k^{-1/2}\boldsymbol{\sigma}_m\|_{0,\Omega}^2$$

for the mixed method and

$$\begin{aligned} \text{err}_\text{nc} &= \|k^{1/2}\nabla_h(u-u_{nc})\|_{0,\Omega}^2 \\ &= \int_{\partial\Omega}(k\nabla u\cdot\mathbf{n})\,u\,ds - 2\sum_{K\in\mathcal{T}}\int_{\partial K}(k\nabla u_{nc}\cdot\mathbf{n})\,u\,ds + \sum_{K\in\mathcal{T}}\|k^{1/2}\nabla u_{nc}\|_{0,K}^2 \end{aligned}$$

for the nonconforming method. Since the true solution u is very smooth near the boundary, the integrations on the boundary can be computed very accurately. We define the so-called effectivity index:

$$\text{eff-index}_{\text{m}} := \frac{\eta}{\|k^{1/2}\nabla u + k^{-1/2}\boldsymbol{\sigma}_m\|_{0,\Omega}} \quad \text{and} \quad \text{eff-index}_{\text{nc}} := \frac{\eta}{\|k^{1/2}\nabla_h(u - u_{nc})\|_{0,\Omega}},$$

and use the following stopping criteria:

$$\text{rel-err}_{\text{m}} := \frac{\|k^{1/2}\nabla u + k^{-1/2}\boldsymbol{\sigma}_m\|_{0,\Omega}}{\|k^{-1/2}\boldsymbol{\sigma}\|_{0,\Omega}} \le \text{tol} \quad \text{and} \quad \text{rel-err}_{\text{nc}} := \frac{\|k^{1/2}\nabla_h(u - u_{nc})\|_{0,\Omega}}{\|k^{1/2}\nabla u\|_{0,\Omega}} \le \text{tol}.$$

Denote by l the number of levels of refinement and by N the number of vertices of triangulation.

The error estimators introduced in [16] for both mixed and nonconforming finite element approximations to the Poisson equations recover the gradient in the continuous linear finite element space. A natural extension of these estimators to the interface problems is to recover either the gradient or the flux again in the continuous linear finite element space. More specifically, for the mixed method, let $\boldsymbol{\sigma}_m$ be the solution of (3.1) and let $\boldsymbol{\rho}_{m,f} \in \mathcal{U}^2$ and $\boldsymbol{\rho}_{m,g} \in \mathcal{U}^2$ satisfy the following problems

$$\begin{aligned} (k^{-1}\boldsymbol{\rho}_{m,f},\, \boldsymbol{\tau}) &= (k^{-1}\boldsymbol{\sigma}_m,\, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathcal{U}^2 \\ \text{and} \quad (k\boldsymbol{\rho}_{m,g},\, \boldsymbol{\tau}) &= -(\boldsymbol{\sigma}_m,\, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathcal{U}^2, \end{aligned}$$

respectively. Then the corresponding error estimators are defined by

$$\eta_{m,CB,f} = \|k^{-1/2}(\boldsymbol{\sigma}_m - \boldsymbol{\rho}_{m,f})\|_{0,\Omega} \quad \text{and} \quad \eta_{m,CB,g} = \|k^{-1/2}\boldsymbol{\sigma}_m + k^{1/2}\boldsymbol{\rho}_{m,g}\|_{0,\Omega}.$$

For the nonconforming method, let u_{nc} be the solution of (3.6) and let $\boldsymbol{\rho}_{nc,f} \in \mathcal{U}^2$ and $\boldsymbol{\rho}_{nc,g} \in \mathcal{U}^2$ satisfy the following problems

$$\begin{aligned} (k^{-1}\boldsymbol{\rho}_{nc,f},\, \boldsymbol{\tau}) &= (-\nabla_h u_{nc},\, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathcal{U}^2 \\ \text{and} \quad (k\boldsymbol{\rho}_{nc,g},\, \boldsymbol{\tau}) &= (k\nabla_h u_{nc},\, \boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathcal{U}^2. \end{aligned}$$

Then the corresponding error estimators are defined by

$$\eta_{nc,CB,f} = \|k^{1/2}\nabla_h u_{nc} + k^{-1/2}\boldsymbol{\rho}_{nc,f}\|_{0,\Omega} \quad \text{and} \quad \eta_{nc,CB,g} = \|k^{1/2}(\nabla_h u_{nc} - \boldsymbol{\rho}_{nc,g})\|_{0,\Omega}.$$

We start with the coarsest triangulation $\mathcal{T}_0$ obtained from halving 16 congruent squares by connecting the bottom left and upper right corners. We report numerical results with the stopping criterion tol = 0.1. From Table 1 and Table 2, Figures 1, 2, 7, and 8, it is clearly that the CB estimators introduce unnecessary refinements along the interfaces. Meshes generated by η_m, $\hat{\eta}_m$, η_{nc}, $\hat{\eta}_{nc}$, and $\bar{\eta}_{nc}$ are similar. By inspecting the effectivity index, all the error estimators introduced in this paper are accurate. Moreover, the slope of the log(dof)-log(relative error) for all estimators is $-1/2$, which indicates the optimal decay of the error with respect to the number of unknowns.

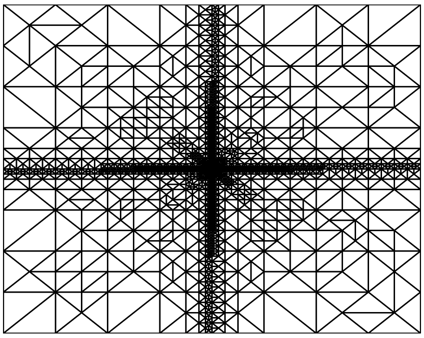

Figure 1: mesh generated by $\eta_{m,CB,f}$

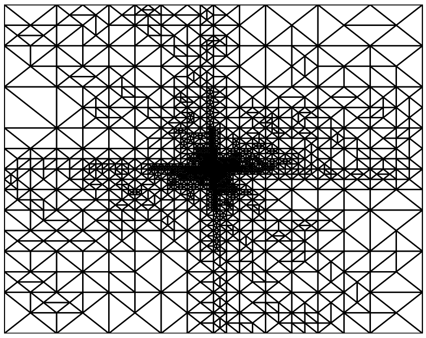

Figure 2: mesh generated by $\eta_{m,CB,g}$

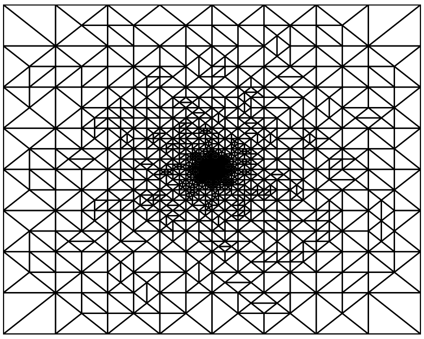

Figure 3: mesh generated by η_m

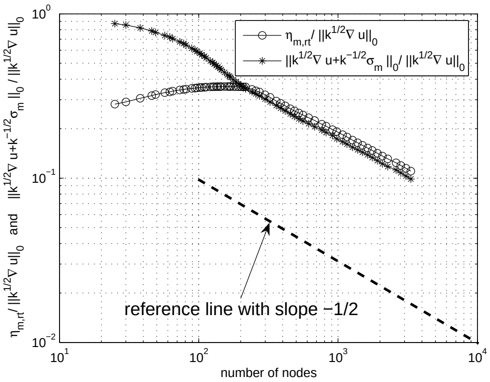

Figure 4: error and estimator η_m

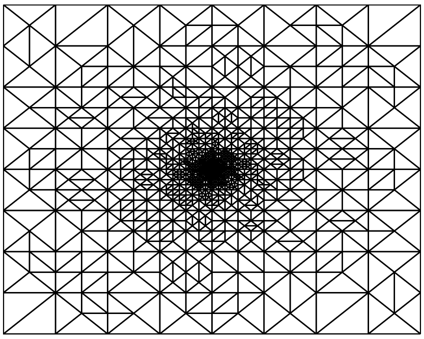

Figure 5: mesh generated by $\hat{\eta}_m$

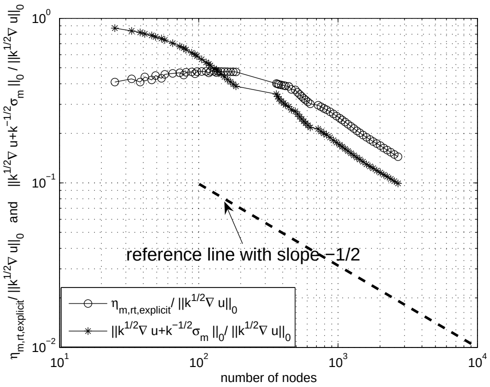

Figure 6: error and estimator $\hat{\eta}_m$

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	l	N	err	rel-err	η	eff-index
η_m	66	3329	0.0558	0.0988	0.0623	1.1170
$\hat{\eta}_m$	69	2693	0.0560	0.0992	0.0815	1.4537
$\eta_{m,CB,f}$	96	7169	0.0557	0.0986	0.0731	1.3110
$\eta_{m,CB,g}$	83	4021	0.0556	0.0984	0.1845	3.3208

Table 1: Comparison of estimators for relative error less than 0.1 for mixed methods

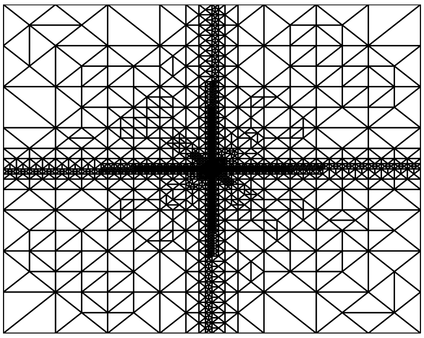

Figure 7: mesh generated by $\eta_{nc,CB,f}$

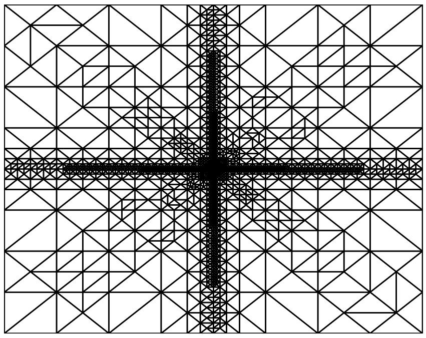

Figure 8: mesh generated by $\eta_{nc,CB,g}$

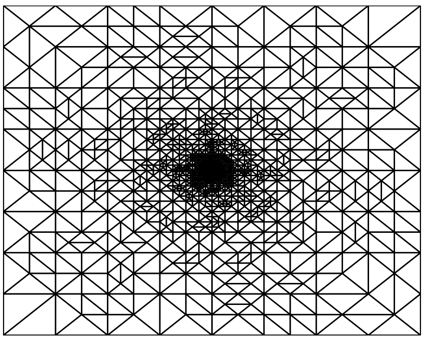

Figure 9: mesh generated by η_{nc}

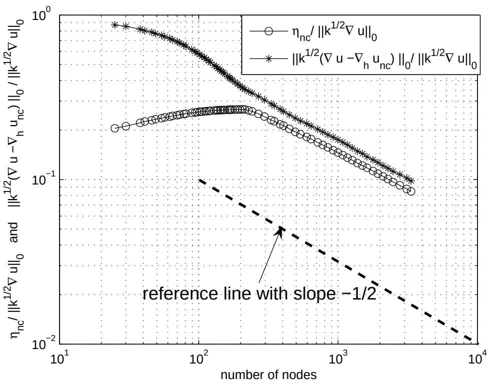

Figure 10: error and estimator η_{nc}

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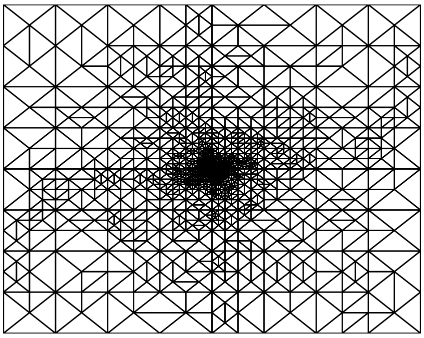

Figure 11: mesh generated by $\hat{\eta}_{nc}$

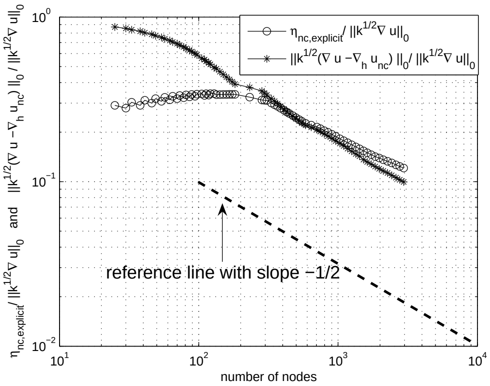

Figure 12: error and estimator $\hat{\eta}_{nc}$

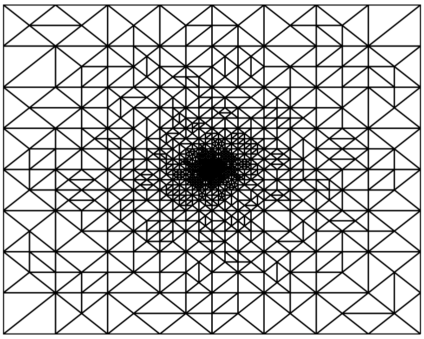

Figure 13: mesh generated by $\bar{\eta}_{nc}$

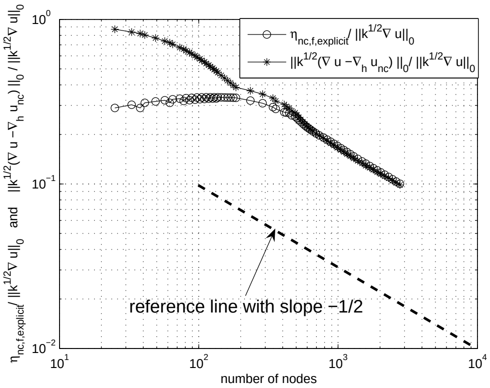

Figure 14: error and estimator $\bar{\eta}_{nc}$

	l	N	err	rel-err	η	eff-index
η_{nc}	70	3343	0.0479	0.0982	0.0479	0.8636
$\bar{\eta}_{nc}$	72	2780	0.0557	0.0985	0.0566	1.0170
$\hat{\eta}_{nc}$	71	2960	0.0563	0.0997	0.0683	1.2133
$\eta_{nc,CB,f}$	97	7090	0.0562	0.0995	0.0736	1.3083
$\eta_{nc,CB,g}$	102	6369	0.0562	0.0995	0.0780	1.3879

Table 2: Comparison of estimators for relative error less than 0.1 for nonconforming methods

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