

14.6 Directional Derivatives

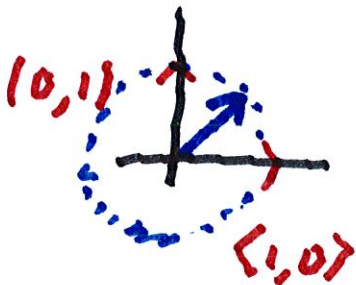
$$\lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} = \text{rate of change}$$

in the $(1, 0)$ direction

and

$$\lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h} = \text{rate of change in the}$$

$(0, 1)$ direction



Note that $(1, 0)$

and $(0, 1)$ are unit vectors.

Now let \vec{U} be any unit vector

We define

$$D_{\vec{U}} f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h}$$

Ex. Suppose $x = x_0 + ha$ and $y = y_0 + hb$

We want to figure out the rate
of change as $h \rightarrow 0$ when $h=0$

We define $g(h) = f(x_0 + ah, y_0 + bh)$

where $\langle a, b \rangle$ is a unit vector.

By the Chain Rule, we get

$$\frac{dg}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$


$$\text{OR } \frac{df}{dt} = \frac{\partial f}{\partial x}(x_0, y_0) a + \frac{\partial f}{\partial y}(x_0, y_0) b$$

$$\frac{df}{dh} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dh} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dh}$$

$$= \frac{\partial f}{\partial x}(x_0, y_0) \cdot a + \frac{\partial f}{\partial y}(x_0, y_0) \cdot b$$

or:

$$D_{\vec{u}} f(x, y) = f_x(x, y) a + f_y(x, y) b$$


 This is called the directional derivative of f at (x, y) in the direction of a unit vector \vec{u}

Ex. Let $f(x, y) = x^2y - y^2 - x^3$

Compute $D_{\vec{v}} f(x_0, y_0)$



at $x_0 = 1, y_0 = 2$ and $\vec{v} = \frac{3}{\sqrt{10}} \vec{i} + \frac{1}{\sqrt{10}} \vec{j}$

$$\frac{\partial f}{\partial x} = 2xy - 3x^2 = 4 - 3 = 1$$

$$\frac{\partial f}{\partial y} = x^2 - 2y = -3$$

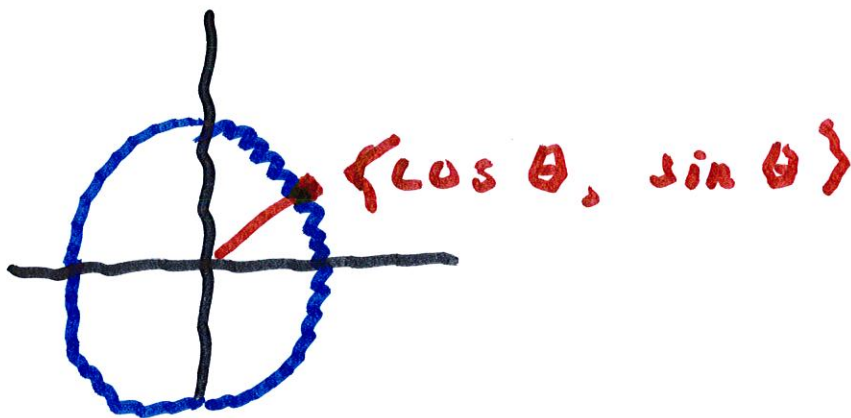
$$D_{\vec{v}} f = 1 \cdot \frac{3}{\sqrt{10}} - 3 \cdot \frac{1}{\sqrt{10}} = 0$$

If we use unit vector \vec{v} that makes an angle of θ with the x-axis, then

$$\frac{dg}{dt} = \frac{\partial f}{\partial x}(x_0, y_0) \cos \theta + \frac{\partial f}{\partial y}(x_0, y_0) \sin \theta$$

is the direction of

$$\vec{u} = \langle \cos \theta, \sin \theta \rangle.$$



Ex. Find the directional

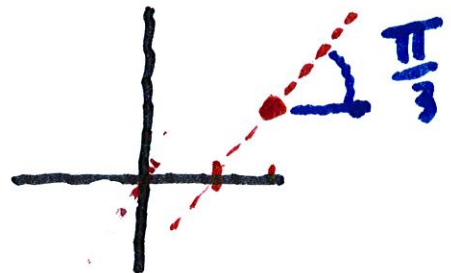
derivative $D_{\vec{u}} f(x, y)$ if

$$f(x, y) = x^2 - xy^3 + y^2 \quad \text{and}$$

\vec{u} is the unit vector pointing

in the direction with angle $\theta = \frac{\pi}{3}$

and $(x_0, y_0) = (2, 1)$.



$$f_x = 2x - y^3 = 4 - 1 = 3$$

and

$$f_y = -3xy^2 + 2y = -6 + 2 = -4$$

$$\text{Since } \vec{v} = \frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j},$$

$$D_{\vec{v}} f(2,1) = 3 \cdot \frac{1}{2} - 4 \cdot \frac{\sqrt{3}}{2}$$

$$= \underline{\underline{\frac{3}{2} - 2\sqrt{3}}}}$$

Note that we can write

$$D_{\vec{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

$$= \langle f_x(x, y), f_y(x, y) \rangle \cdot \vec{u}$$

We call this the gradient of f
at (x, y) .

$$\nabla f(x, y) = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j}$$

Hence $D_{\vec{u}} f(x, y) = \underline{\underline{\nabla f(x, y) \cdot \vec{u}}}$

Ex. Find the directional derivative

of the function $f(x,y) = x^2y^3 - y^2$

at the point $(3, 2)$ in the

direction of $\vec{u} = \frac{2\vec{i} + 3\vec{j}}{\sqrt{13}}$

$$\frac{\partial f}{\partial x} = 2xy^3 = 6 \cdot 8 = 48$$

$$\frac{\partial f}{\partial y} = 3x^2y^2 - 2y = 108 - 4 = 104$$

$$\therefore \nabla f(3,2) = 48\vec{i} + 104\vec{j}$$

$$\Rightarrow D_{\vec{u}}f = 48 \cdot \frac{2}{\sqrt{13}} + 104 \cdot \frac{3}{\sqrt{13}}$$

$$= \frac{408}{\sqrt{13}}$$

Functions of 3 variables:

Given a function $f(x,y,z)$

and a unit vector $\vec{u} = \langle a, b, c \rangle$

at (x_0, y_0, z_0) , then we define

$$D_{\vec{u}} f(x_0, y_0, z_0)$$

$$= \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

We can compute $D_{\vec{u}} f(x, y, z)$

$$= \frac{\partial f}{\partial x}(x, y, z)a + \frac{\partial f}{\partial y}(x, y, z)b + \frac{\partial f}{\partial z}(x, y, z)c$$

We can define the directional

derivative by $\nabla f =$

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

The quantity $\nabla f \cdot \vec{u}$

depends on which vector we choose.

Recall the formula $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$

$$\nabla f \cdot \vec{u} = |\nabla f| |\vec{u}| \cos \theta$$

$$= |\nabla f| \cos \theta$$

We get the largest value when

$\cos \theta = 1$, i.e., when \vec{U} points

in the same direction as ∇f .

But remember, \vec{U} must be a

unit vector. i.e., when

$$\vec{U} = \frac{\nabla f}{|\nabla f|}$$

Then $\nabla f \cdot \vec{u}$

$$= \frac{\nabla f \cdot \nabla f}{|\nabla f|} = \frac{|\nabla f|^2}{|\nabla f|}$$

$$= \underline{\underline{|\nabla f|}}$$

To minimize

$$\text{set } \vec{u} = -\frac{\nabla f}{|\nabla f|}$$

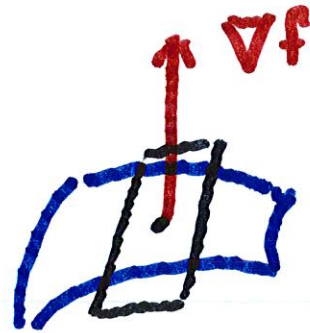
In sum, $\nabla f \cdot \vec{u}$ is maximized

$$\text{when } \vec{u} = \frac{\nabla f}{|\nabla f|}$$

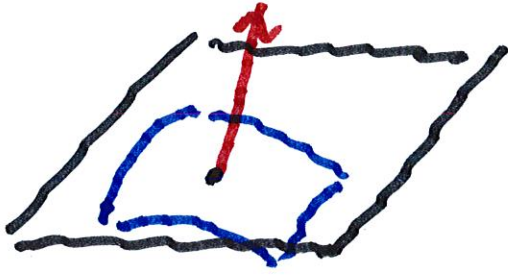
and the largest

$$\text{value is } = \underline{\underline{|\nabla f|}}$$

Tangent Planes to
level surfaces.



Note that if we travel
in the unit direction \vec{u} ,
then the rate of change is $= 0$
if the motion vector \vec{u}
satisfies $\nabla f \cdot \vec{u} = 0$.



We can define the tangent plane by

$$F_x(x_0, y_0, z_0)(x - x_0)$$

(1)

$$+ F_y(x_0, y_0, z_0)(y - y_0)$$

$$+ F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

Ex. Find the tangent planes

to the level surface

$$f(x, y, z) = k$$

through (x_0, y_0, z_0) by using (1).

Ex. Find the equation of the

surface $x^2 - y^2 + 2z^2 + 5 = 0$.

through $(1, 2, -1)$ $F(x, y, z)$

$$F_x = 2x = 2 \cdot 1$$

$$F_y = -2y = -4$$

$$F_z = -4z = 4$$

$$\left. \begin{array}{l} F_x = 2x = 2 \cdot 1 \\ F_y = -2y = -4 \\ F_z = -4z = 4 \end{array} \right\} \text{at } (1, 2, -1)$$

$$\therefore 2(x-1) - 4(y-2) + 4(z+1) = 0$$

Find the slope of the tangent

line of the curve $x^2 - y^3 + 7 = 0$

at $(1, 2)$

A surface $z = z(x, y)$ is

defined by

$$0 = F(x, y, z(x, y)) \text{ at } (x_0, y_0, z_0)$$

Compute $\frac{\partial z}{\partial x}(x, y)$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}(x, y) \frac{\partial z}{\partial x} = 0$$

$$\text{Or: } \frac{\partial z}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

Ex. If $z(x, y)$ satisfies

$$x^2y - yz^2 + z^3 = 1,$$

find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$$\frac{\partial}{\partial x} (x^2y - yz^2 + z^3) = 0$$

$$-2xy - 2yz \frac{\partial z}{\partial x} + 3z^2 \frac{\partial z}{\partial x} = 0$$

$$\rightarrow \frac{\partial z}{\partial x} = \frac{2xy}{3z^2 - 2yz}$$

$$\frac{\partial}{\partial y} (x^2 y - y z^2 + z^3) = 0$$

$$x^2 - z^2 - 2yz \frac{\partial z}{\partial y} + 3z^2 \frac{\partial z}{\partial y} = 0$$

$$\rightarrow \frac{\partial z}{\partial y} = \frac{z^2 - x^2}{3z^2 - 2yz}$$

