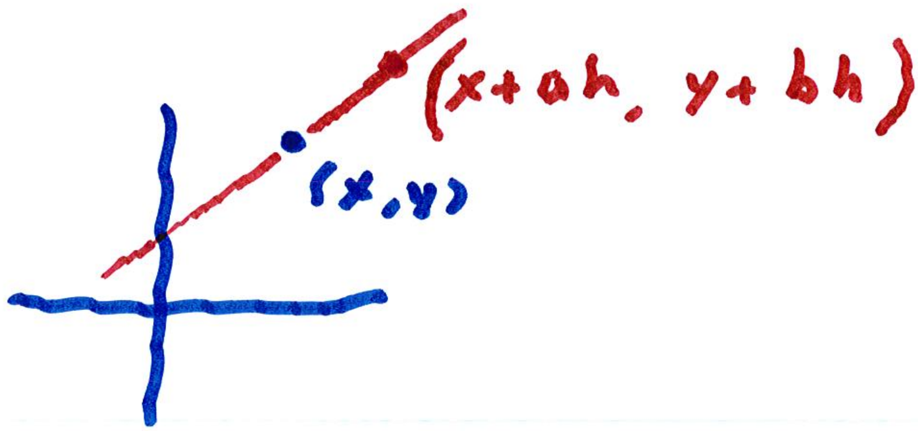
14.6 cont'd.

If $\vec{u} = \langle a, b \rangle$ is a unit vector, and if $f(x,y)$ is a function, then

$$D_{\vec{u}} f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$$

$D_{\vec{u}} f$ = rate of change of

$f(x+ha, y+hb)$ at $h=0$



Ex. Find the directional derivative $D_{\vec{u}} f(x,y)$ if $f(x,y) = x^3 - 2xy + 4y$ and $\vec{u}$ = unit vector given by angle $\theta = \frac{-\pi}{6}$. What is $D_{\vec{u}} f(2,1)$?

$$\nabla f = \langle 3x^2 - 2y \, , \, -2x + 4 \rangle$$

at (2,1) in the direction

$\vec{u}$ that makes an angle of

$-\frac{\pi}{6}$ $\quad \nabla f = \langle 10 \, , \, 0 \rangle \cdot \langle \frac{\sqrt{3}}{2} \, , \, -\frac{1}{2} \rangle$

$$= 5\sqrt{3}$$

Recall that

$$D_{\vec{u}} f(x,y) = \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b$$

$$= \left\langle f_x(x,y),\, f_y(x,y) \right\rangle \cdot \vec{u}$$

$$\left(\text{since } \vec{u} = \langle a, b \rangle \right)$$

If we write

$$\nabla f(x,y) = \left\langle f_x(x,y),\, f_y(x,y) \right\rangle$$

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

This is called the gradient of f at (x,y)

$$\therefore\ D_{\vec{u}} f(x,y) = \nabla f_{(x,y)} \cdot \vec{u}$$

Ex. Use the gradient to compute the dir. deriv. of $f(x,y) = 2x^2y - y^2$ at $(-2, 1)$ in the direction of $\langle -1, 1 \rangle$.

$$\frac{\partial f}{\partial x} = 4xy \qquad \frac{\partial f}{\partial y} = 2x^2 - 2y$$

$$\nabla f = -8\vec{i} + 7\vec{j}$$

Note that $\vec{u} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

$$\therefore \nabla f \cdot \vec{u} = \frac{8}{\sqrt{2}} + \frac{7}{\sqrt{2}} = \frac{15}{\sqrt{2}}$$

The case when $f = f(x, y, z)$

is similar.

If we write $\vec{u} = \langle a, b, c \rangle$

then

$$D_{\vec{u}} f(x, y, z)$$

$$= \lim_{h \to 0} \frac{f(x + ha, y + hb, z + hc) - f(x, y, z)}{h}$$

Using vector notation,

with $\vec{X}_0 = \langle x, y, z \rangle$ and

$\vec{u} = \langle a, b, c \rangle$, we get

$$D_{\vec{u}} f(\vec{x}_0) = \lim_{h \to 0} \frac{f(\vec{x}_0 + t\,\vec{u}) - f(\vec{x}_0)}{h}$$

Similarly, we can define

gradient of f by

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

Ex. Find the dir.-der of

$f(x, y, z) = xy + yz + xz$

at $(1, -1, 3)$ in the direction of

$(2, 4, 5)$.

$$\nabla f(x, y, z) = \langle y+z,\ x+z,\ x+y \rangle$$

$$= \langle 2,\ 4,\ 0 \rangle$$

Note that $\vec{u} = \left\langle \frac{2}{\sqrt{45}}, \frac{4}{\sqrt{45}}, \frac{5}{\sqrt{45}} \right\rangle$

$$\therefore D_{\vec{u}} f = \frac{4}{\sqrt{45}} + \frac{16}{\sqrt{45}} = \frac{20}{\sqrt{45}}$$

Thm. Suppose f is a differentiable function of 2 or 3 variables.

The maximal value of the directional derivative $D_{\vec{u}} f(\vec{x})$ is $|\nabla f(\vec{x})|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(\vec{x})$.

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f| |\vec{u}| \cos\theta$$

$$= |\nabla f| \cos\theta$$

Max. Val. of $\cos\theta$ is when

$\theta = 0$. $\theta = 0 \leftrightarrow$ [illegible] , which

means $\vec{u} = \dfrac{\nabla f}{|\nabla f|}$

Ex. Find the unit vector that gives the largest value of $D_{\vec{u}} f$, when $f(x,y,z) = xe^{yz}$

$f_x = e^{yz}$, $f_y = xze^{yz}$, $f_z = xye^{yz}$

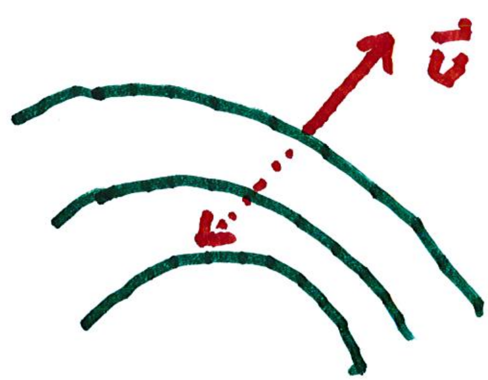
$\nabla f = e^{yz}\langle 1, xz, xy \rangle$

Dir. of maximal increase is

$$\frac{(1, xz, xy)}{\sqrt{1 + x^2z^2 + x^2y^2}}$$

The direction of maximal decrease occurs when $\cos\theta = -1$, where θ is the angle between $\vec{u} = \dfrac{\nabla f}{|\nabla f|}$

Thus $\vec{u} = -\dfrac{\nabla f}{|\nabla f|}$



Suppose S is a surface with equation $F(x, y, z)$. Let C be any curve which lies on S and which is defined by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Hence $F(x(t), y(t), z(t))$

If we differentiate in t,

we get

$$\frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0$$

Since $\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle$

and $\nabla F = \langle F_x, F_y, F_z \rangle$,

we get

$$\nabla F \cdot \vec{r}\,'(t) = 0$$

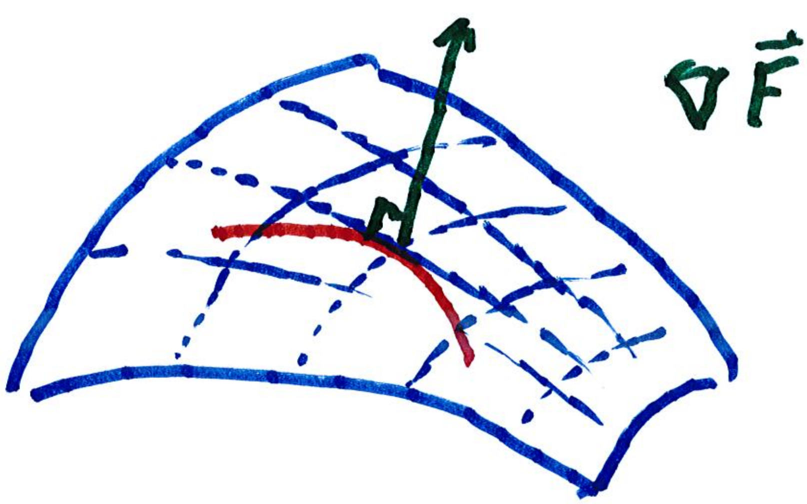
This shows us that $\nabla f(x_0, y_0, z_0)$

is $\perp$ to the tangent vector

vector $\vec{r}\,'(t_0)$ to any curve

C on S that passes

through (x_0, y_0, z_0)



$\therefore$ ∇F is the vector that

is $\perp$ to the plane through

(x_0, y_0, z_0),

we get

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0)$$

$$+ F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

The normal line to S at P

is $\perp$ to the tangent plane,

and satisfies

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

In the special case when

the equation of the form

$z = f(x,y)$, we can write

$$F(x,y,z) = f(x,y) - z = 0,$$

then $F_x(x,y,z) = f_x(x,y)$

$$F_y(x,y,z) = f_y(x,y)$$

$$F_z(x,y,z) = -1$$

$\therefore$ The tangent plane can

be described by

$$f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0)$$
$$- (z-z_0) = 0$$

This shows that this formula

agrees with the more general

formula given by

$$F_x(\vec{x}_0)(x-x_0) + F_y(\vec{x}_0)(y-y_0)$$
$$+ F_z(\vec{x}_0)(z-z_0) = 0$$

Ex. Suppose an ellipsoid S

is described by

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

Therefore

$F_x(x,y,z) = \frac{x}{2}$ $F_y(x,y,z) = 2y$ and

$F_z(x,y,z) = \frac{2z}{9}$

$\therefore F_x(-2,1,-3) = -1$ $F_y(-2,1,-3) = 2$

$F_z(-2,1,-3) = -\frac{2}{3}$

This gives the equation of the tangent plane at $(-2, 1, -3)$ as

$$-1(x+2) + 2(y-1) - \frac{2}{3}(z+3) = 0$$