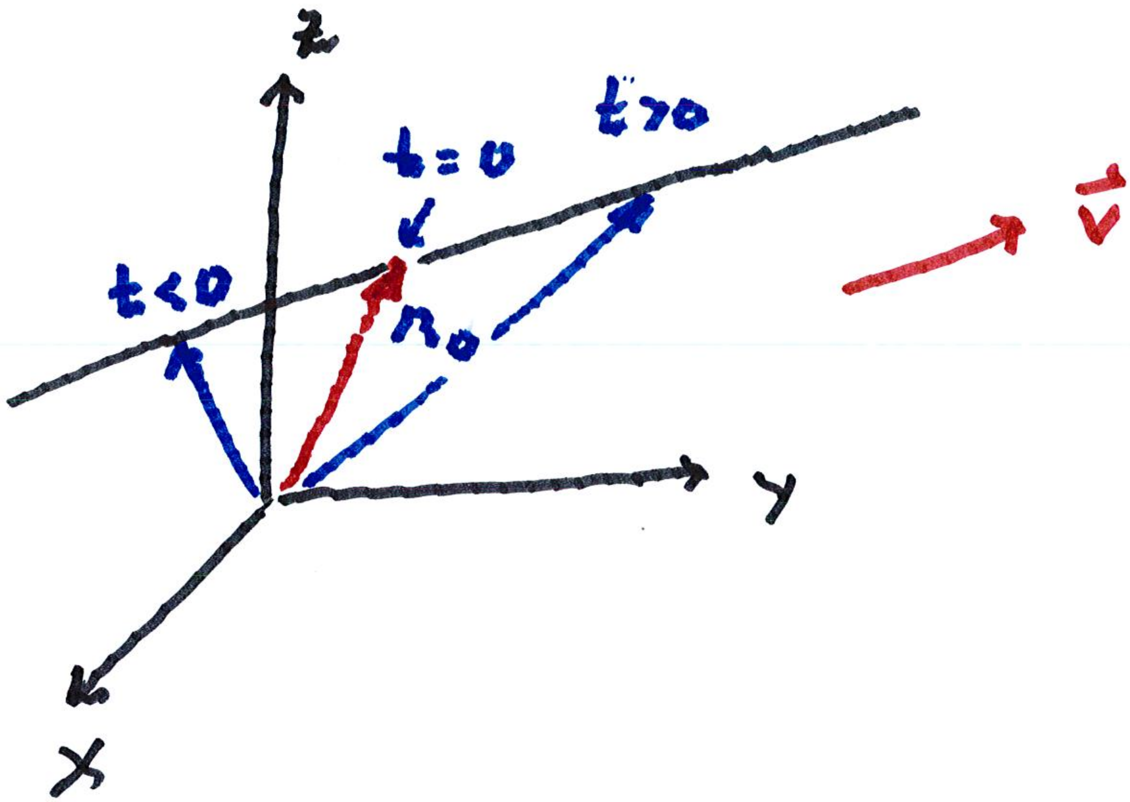
12.5 Lines and Planes in Space

Any line L is determined by a point $\vec{r}_0$ in L and a vector $\vec{v}$ that is parallel to L. Then L is

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

This is the vector equation of L.



If $\vec{r}_0 = \langle x_0 , y_0 , z_0 \rangle$

and $r(t) = \langle x(t), y(t), z(t) \rangle$

$= \langle x , y , z \rangle$

and if $\vec{v} = \langle a, b, c \rangle$, then

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

or

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

This is called the

parametric equation of L.

Find the vector equation

and parametric equations

of the line L through $(2, 1, -1)$

that is parallel to $\langle 1, 2, 3 \rangle$

Set $\vec{r}_0 = \langle 2, 1, -1 \rangle$ and

$\vec{v} = \langle 1, 2, 3 \rangle$

$\vec{r} = \langle 2, 1, -1 \rangle + t\langle 1, 2, 3 \rangle$

(vector equation)

and

$x = 2 + t, \quad y = 1 + 2t, \quad z = -1 + 3t$

(parametric equations)

Another way of describing L is to eliminate the parameter t:

Solving for t in all 3 parametric equations.

$$\frac{x - x_0}{a} = t \qquad \frac{y - y_0}{b} = t \qquad \frac{z - z_0}{c} = t$$

Since all equations are equal,

we get

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These are called the

symmetric equations of L

Ex. Find the vector equation the parametric equations and the symmetric equations of the line L that passes through $\langle 2, 3, -1 \rangle$ and $\langle 1, 1, 2 \rangle$.

$$\vec{v} = \langle 1, 1, 2 \rangle - \langle 2, 3, -1 \rangle$$

$$\vec{v} = \langle -1, -2, 3 \rangle$$

Also, set $\vec{r}_0 = \langle 2, 3, -1 \rangle$

vec. eqn:

$$\vec{r} = \langle 2, 3, -1 \rangle + t \langle -1, -2, 3 \rangle$$

par. eqns

(1) $x = 2 - t$, $y = 3 - 2t$, $z = -1 + 3t$

sym. eq'ns:

$$\frac{x-2}{-1} = \frac{y-3}{-2} = \frac{z+1}{3}$$

Ex. At what point does the above line L (in (11)) pass through the plane $x - y + 2z = 1$.

We substitute in the parametric equations of L into the equation of the plane.

$$(2-t) - (3-2t) + 2(-1+3t) = 1$$

$$\to (2-3-2) + 7t = 1 \to t = \frac{4}{7}$$

$\therefore$ point is $\left\langle \frac{10}{7}, \frac{13}{7}, \frac{5}{7} \right\rangle$.

Two lines L_1 and L_2 are skew if they are not parallel and don't intersect. Show that

L_1 : $x = 2+t$, $y = 1-2t$, $z = 3+3t$

L_2 : $x = 3-t$, $4-4t$, $z = 1+2t$

are skew:

First, the direction vectors

$\langle 1, -2, 3 \rangle$ and $\langle -1, -4, 2 \rangle$

are not parallel.

We write L_2 as

$x = 3 - s, \quad y = 4 - 4s, \quad z = 1 + 2s$

A point of intersection

would satisy

$$3 - s = 2 + t \qquad E_1$$

$$4 - 4s = 1 - 2t \qquad E_2$$

$$1 + 2s = 3 + 3t \qquad E_3$$

Now eliminate t in E_1, E_2

$2E_1 + E_2$: $2(3-s) + (4-4s) = 5$

or $10 - 6s = 5$

or $s = \dfrac{5}{6}$

Now eliminate t in E_2, E_3

$3E_2 + 2E_3$: $3(4-4s) + 2(1+2s) = 9$

or $14 - 8s = 9$

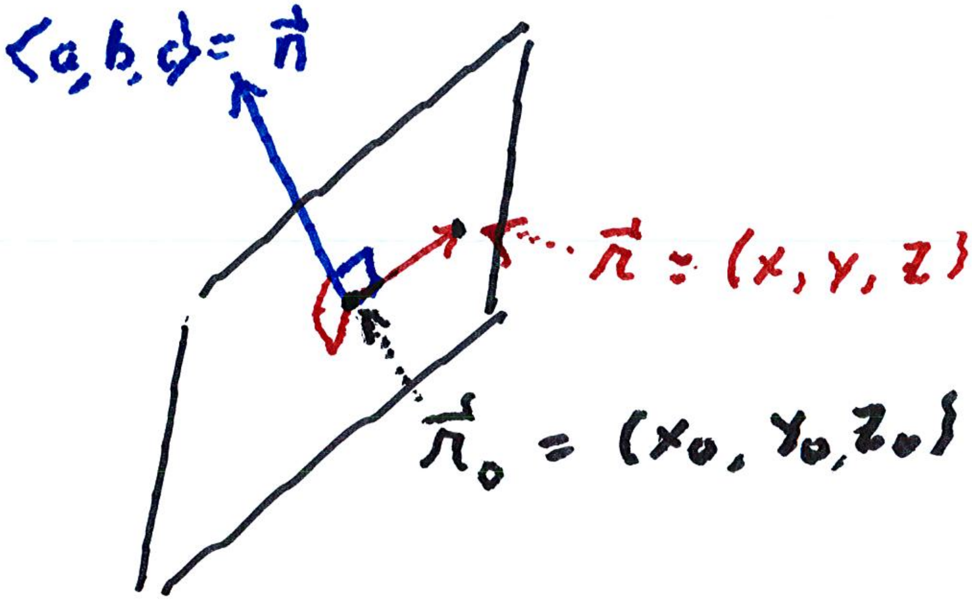
or $s = \frac{5}{8}$

$\therefore$ Lines don't intersect.

$\therefore$ Lines are skew

Planes

A plane $\mathcal{P}$ is determined by a point $\vec{r}_0$ in the plane and a normal vector $\vec{n}$ that is orthogonal (perpendicular) to $\mathcal{P}$. If $\vec{r}$ is any point in $\mathcal{P}$, then $\vec{r} - \vec{r}_0$ lies in $\mathcal{P}$.



$$(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$$

$$0 = \langle x - x_0, \ y - y_0, \ z - z_0\rangle \cdot \langle a, b, c\rangle$$

and so, $\vec{n}$ is $\perp$ to $\vec{r} - \vec{r}_0$

$$\therefore \quad \vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \qquad (*)$$

$$\text{or} \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0$$

holds for every r in P.

This is called the

vector equation of the plane

If $\vec{n} = \langle a, b, c \rangle$

and $\vec{r} = \langle x, y, z \rangle$ and

$\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$,

then $(*)$ becomes

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

or $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

This is called the <u>scalar equation</u> of $\mathcal{P}$

Ex. Find an equation of the plane P through the point $(1, 4, 2)$ with normal vector

$r = \langle x, y, z \rangle$

$\vec{n} = \langle 3, 2, 5 \rangle$. The scalar eq'n is

$$3(x-1) + 2(y-4) + 5(z-2) = 0$$

or $$3x + 2y + 5z = 3 + 8 + 10$$

or $$3x + 2y + 5z = 21$$

This last equation is

called a linear equation of θ.

Ex. Find the linear equation of the plane P that passes through $P(1, 2, -2)$, $Q(2, 1, 1)$ and $R(1, 3, 4)$.

Note that $\overrightarrow{PQ} = \langle 2-1, 1-2, 1-(-2) \rangle$

$= \langle 1, -1, 3 \rangle$ is in P.

$\vec{n} \nearrow$

So does $\overrightarrow{PR} = \langle 1-1, 3-2, 4-(-2) \rangle$

$= \langle 0, 1, 6 \rangle = \vec{b}$

Hence $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 3 \\ 0 & 1 & 6 \end{vmatrix}$

$= (-6-3)\vec{i} - (6-0)\vec{j} + (1-0)\vec{k}$

$= -9\vec{i} - 6\vec{j} + \vec{k}$ is $\perp$

to P.

Set $\vec{n} = \langle -9, -6, 1 \rangle$

and $\vec{r}_0 = \langle 1, 2, -2 \rangle$

The vector equation is

$\vec{n} \cdot (\vec{r} - \vec{r}_0)$ or

$$-9(x-1) - 6(y-2) + (z+2) = 0$$

or $\underline{-9x - 6y + z = -9 - 12 - 2 = -23}$