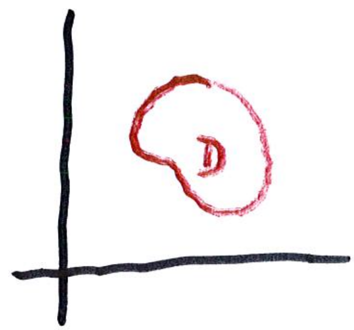
15.5 Density and Mass

Consider a thin plate in the shape of

a domain D



The density of

D may vary

as a function of (x,y). We define

$$\rho(x,y) = \lim \frac{\Delta m}{\Delta A} \rightarrow \Delta m \approx \rho(x,y)\Delta A$$

In the usual way, we can

approximate D by a union of

rectangles $R_{i,j}$, each of area

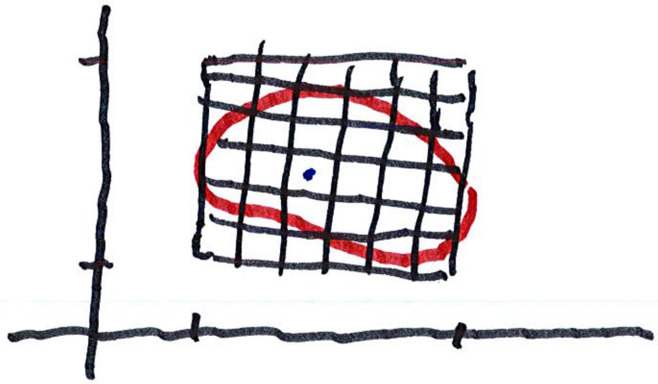
$\Delta A = \Delta x \, \Delta y$. The mass of

the rectangle R_{ij} is approx.

$= \rho(x_i, y_j) \, \Delta x \, \Delta y.$

$\therefore$ The total mass is approx

$$m \approx \sum_{i,j} \rho(x_i, y_j) \, \Delta x \, \Delta y$$



So $m = \iint_D \rho(x,y)\, dA$

Similarly, if $\sigma(x,y)$ = charge density at (x,y), then the total charge is

$$Q = \iint_D \sigma(x,y)\, dA$$

Moments and Center of Mass

In Chapter 8, we defined

Moment of D about y-axis $= \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\, \rho(x_i, y_j) \underbrace{}_{dA}$

$$\to \iint_D x\, \rho(x,y)\, dA$$

$$= M_y$$

and

Mom. of D about x-axis $= \displaystyle\iint_D y\, \rho(x,y)\, dA$

$$= M_x$$

As before, the center of mass of

D is at $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m},$$

where $m = \iint_D \rho(x,y)\, dA$.

Ex. Find the center of mass of D if

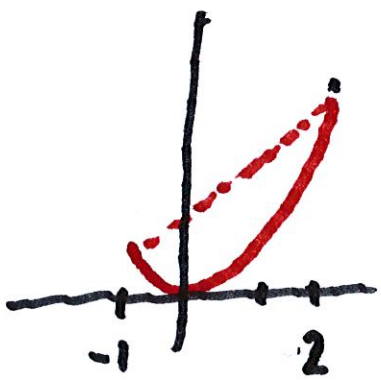
D is bounded by: $y = x^2$ and $y = x+2$

and $\rho(x,y) = y$.

$y = x^2$ and $y = x+2$ intersect

whr $x^2 = x+2$ or $x^2 - x - 2 = 0$

$\rightarrow (x-2)(x+1) = 0 \rightarrow x = -1$ or $x = 2$



$$\text{Mass} = \iint_D y \, dA$$

$$= \int_{-1}^{2} \int_{x^2}^{x+2} y \, dy \, dx$$

$$= \int_{-1}^{2} \frac{y^2}{2} \Big|_{x^2}^{x+2} dx = \int_{-1}^{2} \left(\frac{(x+2)^2}{2} - \frac{x^4}{2} \right) dx$$

$m =$

$$= \int_{-1}^{2} -\frac{x^4}{2} + \frac{x^2}{2} + 2x + 2 \quad dx$$

$$= -\frac{x^5}{10} + \frac{x^3}{6} + x^2 + 2x \Big|_{-1}^{2} = \text{mass}$$

$$\bar{x} = \frac{M_y}{m}, \quad \text{where}$$

$$M_y = \iint_D xy \, dA$$

$$= \int_{1}^{2} \int_{x^2}^{x+2} xy \, dy \, dx$$

$$= \int_{-1}^{2} \frac{xy^2}{2} \Big|_{y=x^2}^{y=x+2} dx$$

$$= \frac{1}{2} \int_{-1}^{2} -x^5 + x^3 + 4x^2 + 4x \quad dx$$

$$= \frac{1}{2} \left[-\frac{x^6}{6} + \frac{x^4}{4} + \frac{4x^3}{3} + 2x^2 \right] \Bigg|_{-1}^{2}$$

$$= M_y$$

Now $\bar{y} = \dfrac{M_x}{m}$, where

$$M_x = \iint_D y^2 \, dA$$

$$= \int_{-1}^{2} \frac{y^3}{3} \Bigg|_{y=x^2}^{y=x+2} dx$$

$$= \int_{-1}^{2} \frac{(x+2)^3}{3} - x^6 \, dx$$

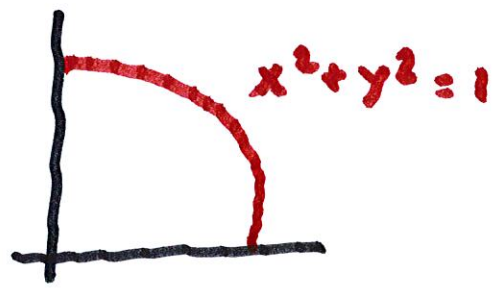
$$= \int_{-1}^{2} -x^6 + \frac{x^3}{3} + 2x^2 + 4x + \frac{8}{3} \, dx$$

$$= \frac{-x^7}{7} + \frac{x^4}{12} + \frac{2x^3}{3} + 2x^2 + \frac{8x}{3} \Bigg|_{-1}^{2}$$

$$\overset{\shortparallel}{M_x}$$

$$\therefore \ \bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$$

Ex. Find $(\bar{x}, \bar{y})$ if D = region in first ~~octant~~ quadrant bounded by $x^2 + y^2 = 1$ and where the density $= y$



$$\bar{x} = \frac{M_y}{m}$$

$$m = \iint_D y \, dA = \int_0^{\pi/2} \int_0^1 r \sin\theta \; r \, dr \, d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \sin\theta \, d\theta$$

$$= -\frac{1}{3} \cos\theta \Big|_0^{\pi/2} = \frac{1}{3}$$

$$M_y = \iint_D xy \, dA$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 r\cos\theta \, r\sin\theta \, r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \frac{\sin\theta\cos\theta}{4} \, d\theta$$

$$= \frac{1}{4}\frac{\sin^2\theta}{2}\Big|_0^{\frac{\pi}{2}} = \frac{1}{8}$$

$$\therefore \bar{x} = \frac{\frac{1}{8}}{\frac{1}{3}} = \frac{3}{8}$$

$$M_x = \iint_D y \cdot y \, dA$$

$$= \int_0^{\pi/2}\int_0^1 (r\sin\theta)^2 \, r \, dr \, d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} \sin^2\theta \, d\theta$$

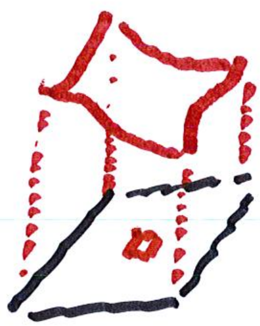
$$= \frac{1}{8} \int_0^{\frac{\pi}{2}} 1 - \cos 2\theta \, d\theta$$

$$= \frac{\theta}{8} - \frac{\sin 2\theta}{16} \Big|_0^{\frac{\pi}{2}}$$

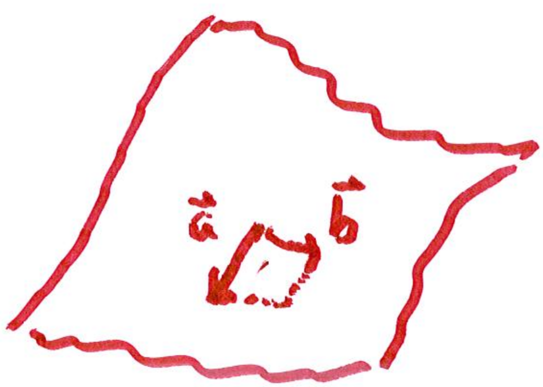
$$= \frac{\pi}{16}$$

$$\therefore \bar{y} = \frac{M_x}{m} = \frac{\pi}{16} \cdot 3 = \frac{3\pi}{16}$$

15.6 Surface Area



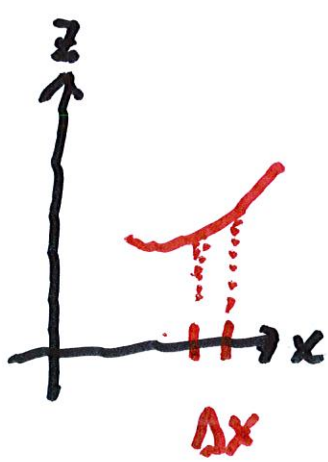
The surface $z = f(x, y)$



$\vec{a} = \Delta x \, \vec{i} + f_x \Delta x \, \vec{k}$

$\vec{b} = \Delta y \, \vec{j} + f_y \Delta y \, \vec{k}$

The vector gen. by $\vec{a} \Delta x \times$



$(\Delta x, f'(x)\Delta x) = \vec{a}$

Compute area of parallelogram

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \Delta x & 0 & f_x \Delta x \\ 0 & \Delta y & f_y \Delta y \end{vmatrix}$$

$$= -f_x \Delta x \Delta y \vec{i} - f_y \Delta x \Delta y \vec{j} + \Delta x \Delta y$$

$\vec{k}$

$$|\text{Area}| = \left((f_x)^2 + (f_y)^2 + 1 \right)^{\frac{1}{2}} \Delta x \Delta y$$

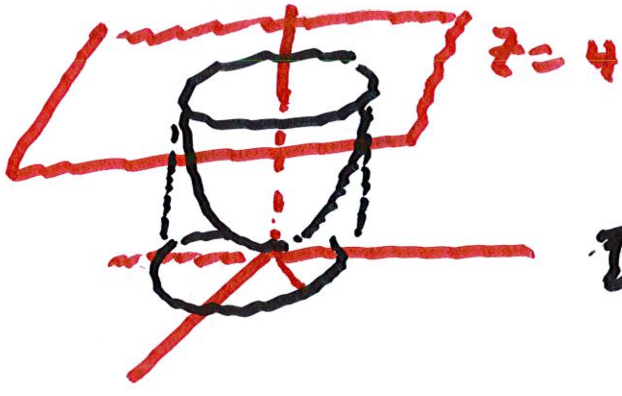
$\therefore$ Area of S above D is

$$A = \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA$$

Ex. Find surface area of

paraboloid $z = x^2 + y^2$ below $z = 4$

$$x^2 + y^2 = z = 4 \rightarrow r = 2$$



$D = D_2(0)$

$$z = x^2 + y^2 = f(x,y)$$

$$f_x = 2x \quad f_y = 2y \qquad \iint_D \sqrt{1+4x^2+4y^2}\, dA$$

$$\rightarrow \sqrt{1+4x^2+4y^2}$$

$$SA = \iint_D \sqrt{1+4(x^2+y^2)}\; dA$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{1+4r^2}\; r\, dr\, d\theta$$

$$= \frac{2\pi}{8} \int_0^2 \sqrt{1+4r^2}\ 8r\,dr$$

$u = 1 + 4r^2$

$du = 8r\,dr$

$$= \frac{\pi}{4} \int_1^{17} \sqrt{u}\ du$$

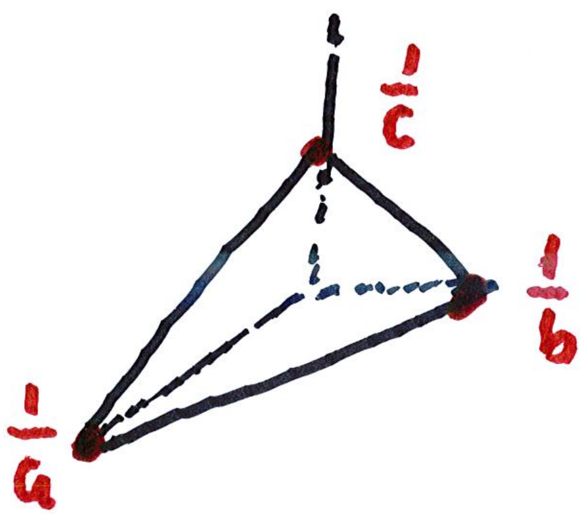
$$= \frac{\pi}{4} \cdot \frac{2}{3} (u)^{3/2} \Big|_1^{17}$$

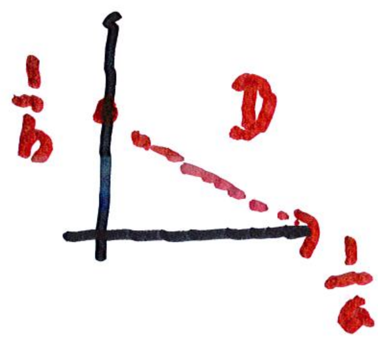
$$= \frac{\pi}{6} \left(17^{3/2} - 1 \right)$$

Ex. Find surface area of

$ax + by + cz = 1$ above

the xy-plane in 1st octant





$$z = \frac{1 - ax - by}{c}$$

$$\frac{\partial f}{\partial x} = -\frac{a}{c} \qquad \frac{\partial f}{\partial y} = -\frac{b}{c}$$

$$A = \iint_D \sqrt{1 + \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2}\, dx\, dy$$

$$= \frac{\sqrt{c^2 + a^2 + b^2}}{c} \iint_D 1 \cdot dA$$

$$\downarrow$$

$$= \frac{1}{2ab}$$

$$= \frac{\sqrt{a^2+b^2+c^2}}{c} \cdot \frac{1}{2ab}$$

$$= \frac{\sqrt{a^2+b^2+c^2}}{2abc}$$