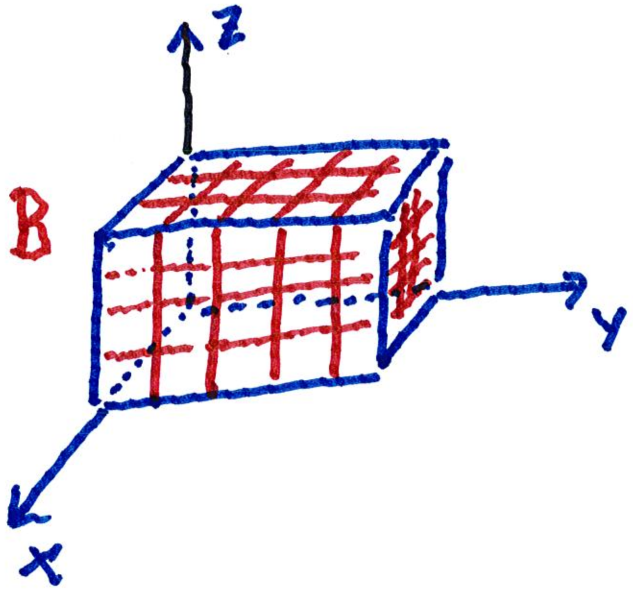
15.7 Triple Integrals

Given a box $B = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} a \le x \le b \\ c \le y \le d \\ r \le z \le s \end{array} \right\}$,

we divide B into sub-boxes

of width Δx, Δy, and Δz,

where $\Delta x = \dfrac{b-a}{l}$, $\Delta y = \dfrac{d-c}{m}$, $\Delta z = \dfrac{s-r}{n}$



We define the triple Riemann sum by

$$\sum_{i=1} \sum_{j=1} \sum_{k=1} f(x_i, y_j, z_k) \Delta x \Delta y \Delta z,$$

where (x_i, y_j, z_k) is in the box

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

If we let l, m, n

$$\iiint_B f(x,y,z)\,dV = \text{limit of Riemann sum as } l, m, n \to \infty$$

This is called the triple integral of f over the box B. To calculate it, we use:

Fubini's Thm.

$$\iiint_B f(x,y,z) = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Integrate first with respect to x, then y, then z.

Or one could integrate first with respect to y, then z, then x, etc.

$$= \int_a^b \int_r^s \int_c^d f(x,y,z)\, dy\, dz\, dx \,, \quad \text{etc.}$$

Ex. If $B = \left\{ (x,y,z) \;\middle|\; 0 \le x \le 2,\ 1 \le y \le 2,\ 0 \le z \le 1 \right\}$

calculate $\displaystyle\iiint_B xy^2 z \; dz\, dy\, dx$

$$= \int_0^2 \int_1^2 \int_0^1 xy^2z \, dz \, dy \, dx$$

$$= \int_0^2 \int_1^2 \frac{xy^2z^2}{2} \Big|_0^1 \, dy \, dx$$

$$= \int_0^2 \int_1^2 \frac{xy^2}{2} \, dy \, dx$$

$$= \int_0^2 \frac{xy^3}{6} \Big|_{y=1}^{y=2} \, dx$$

$$= \int_0^2 \left(\frac{x \cdot 4}{3} - \frac{x}{6} \right) dx = \int_0^2 \frac{7x}{6} \, dx$$

$$\frac{7x^2}{12}\Big|_0^2 = \frac{7}{3}$$

A solid region E is said to be of type 1, if it lies between the graphs of 2 functions of x and y over a region D:

$$\therefore E = \left\{ (x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y) \right\}$$

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{U_1(x,y)}^{U_2(x,y)} f(x,y,z)\,dz \right] dA$$

If D is a type II region, then

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{U_1(x,y)}^{U_2(x,y)} f(x,y,z)\,dz$$

dx dy

or if D is of type I :

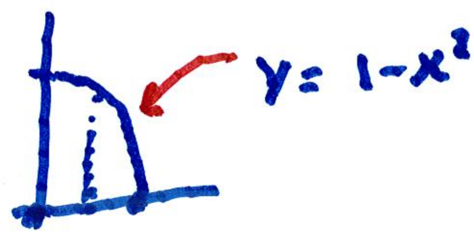
$$\iiint f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{U_1(x,y)}^{U_2(x,y)} f(x,y,z)$$

dz dy dx

Ex. Let D = solid region bounded by $z = x^2 + 2y + 1$ and $z = y + 2$ in the first octant. Find Vol.

$$x^2 + 2y + 1 = y + 2$$

$$\rightarrow \quad y = 1 - x^2$$



$$\text{Vol} = \iiint 1 \, dV$$

$$= \int_0^1 \int_0^{1-x^2} \int_{x^2+2y+1}^{y+2} 1 \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x^2} z \Big|_{x^2+2y+1}^{y+2} \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x^2} (y+2)-(x^2+2y+1) \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x^2} (1-x^2-y) \, dy \, dx$$

$$= \cancel{\int_0^1 \left[y - x^2 y - \frac{y^2}{2} \right]_0^{1-x^2}}$$

$$= \int_0^1 y(1-x^2) - \frac{y^2}{2} \Big|_0^{1-x^2} dx$$

$$= \int_0^1 (1-x^2)^2 - \frac{(1-x^2)^2}{2} \, dx$$

$$= \int_0^1 \frac{1}{2} (1-x^2)^2 \, dx$$

$$= \frac{1}{2} \int_0^1 (1 - 2x^2 + x^4) \, dx$$

$$= \frac{1}{2} \left(x - \frac{2x^3}{3} + \frac{x^5}{5} \right) \Big|_0^1 = \frac{4}{15}$$

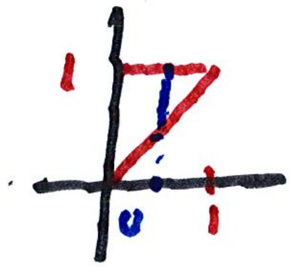
Ex. Let R = triangular region in the xy-plane between $y=x$ and $y=1$ for $0 \le x \le 1$,

and let E = solid region between the surfaces $z=-2$ and $z=1-x^2$ for $(x,y) \in R$

Evaluate $\iint_R \int_{-2}^{1-x^2} (x+1)\, dz\, dy\, dx$

Evaluate $\iiint_E (x+1)\, dV.$

$$= \int_0^1 \int_x^1 \int_{-2}^{1-x^2} (x+1)\, dz\, dy\, dx$$



$$= \int_0^1 \int_x^1 (x+1) z \Big|_{-2}^{1-x^2} dy\, dx$$

$$= \int_0^1 \int_x^1 (x+1) \left\{ 1 - x^2 - (-2) \right\} dy\, dx$$

$$= \int_0^1 \int_x^1 (-x^3 - x^2 + 3x + 3)\, dy\, dx$$

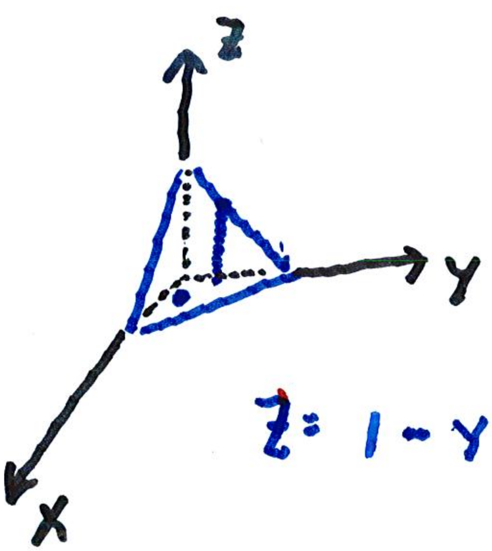
$$= \int_0^1 (-x^3 - x^2 + 3x + 3)\, y \Big|_x^1$$

$$= \int_0^1 (x^4 - 4x^2 + 3)\, dx = \frac{28}{15}$$

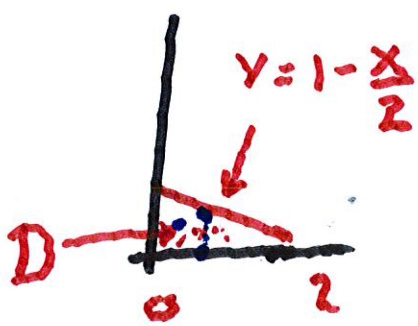
Ex. Suppose that a tetrahedron is bounded by $x + 2y + 2z = 2$, $x = 0$, $y = 0$ and $z = 0$.

and that the density is $\rho = 2z$. Calculate mass m

Set $z = 0 \rightarrow x + 2y = 2 \rightarrow y = 1 - \frac{x}{2}$



$z = 1 - y - \frac{x}{2}$)



$$= \int_0^2 \int_0^{1-\frac{x}{2}} \int_0^{1-y-\frac{x}{2}} 2z \, dz \, dy \, dx$$

$$= \int_0^2 \int_0^{1-\frac{x}{2}} \left(y + \frac{x}{2} - 1 \right)^2 dy \, dx$$

$$= \int_0^2 \left. \frac{\left(y + \frac{x}{2} - 1 \right)^3}{3} \right|_{y=0}^{y=1-\frac{x}{2}} dx$$

$$= \int_0^2 \frac{\left(1 - x/2 + \frac{x}{2} - 1\right)^3}{3} - \frac{\left(\frac{x}{2} - 1\right)^3}{3} \, dx$$

$$= \frac{1}{3}\int_0^2 \left(1 - \frac{x}{2}\right)^3 dx$$

$$= \frac{1}{3}\int_0^2 1 - \frac{3x}{2} + 3\frac{x^2}{4} - \frac{x^3}{8} \, dx$$

$$= \frac{1}{3}\left(x - \frac{3x^2}{4} + \frac{x^3}{4} - \frac{x^4}{32}\right)\Bigg|_0^2$$

$$= \frac{2 - 3 + 2 - \frac{1}{2}}{3} = \underline{\underline{\frac{1}{6}}}$$

Various Applications

$$\text{Vol}(E) = \iiint_E dV = \iiint_E 1\,dV$$

Center of Mass $= (\bar{x}, \bar{y}, \bar{z})$,

where $\dfrac{M_{yz}}{m} = \bar{x}$, $\quad \dfrac{M_{xz}}{m} = \bar{y}$,

and $\dfrac{M_{xy}}{m} = \bar{z}$

where $M_{yz} = \iiint x \rho \, dV$

$M_{xz} = \iiint_E y \rho \, dV$, and

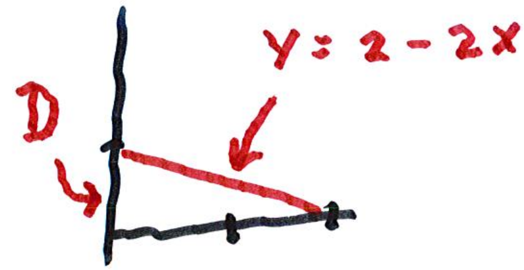
$M_{xy} = \cancel{\iiint_E} = \iiint_E z \rho \, dV$

Ex. Suppose that E = solid tetrahedron bounded by $2x + y + 2z = 2$, $x = 0$, $y = 0$, and $z = 0$. Suppose that the density is $= x$. Calculate mass.

Set $z = 0 \to 2x + y = 2$

$$m = \int_{0}^{2} \int_{0}^{2-2x} \int_{0}^{\frac{2-2x-y}{2}} x \, dz \, dy \, dx$$

$x \quad y \quad z$



Solve for z

$$z = \frac{2 - 2x - y}{2}$$

$$m = \int_0^2 \int_0^{2-2x} xz \Big|_0^{1-x-y/2}$$

$$= \int_0^2 \int_0^{2-2x} x\left(1-x-\frac{y}{2}\right) dy\, dx$$

$$= \int_0^2 \int_0^{2-2x} (x-x^2) - \frac{xy}{2} \quad dy\, dx$$

$$= \int_0^2 (2-2x)(x-x^2)\, dx - \int_0^2 \frac{xy^2}{4}\Big|_{y=0}^{y=2-2x} dx$$

$$= \int_0^2 (2x^3-4x^2+2x)\, dx - \int_0^2 \frac{xy^3}{12}\Big|_0^{2-2x} dx$$

$$= \int_0^2 (2-2x)(x-x^2)\,dx - \int_0^2 \frac{xy^2}{4}\Big|_{y=0}^{y=2-2x} dx$$

$$= \int_0^2 (2x - 4x^2 + 2x^3)\,dx - \int_0^2 \frac{x(2-2x)^2}{4}$$

$$= 4 - \frac{32}{3} + 8 - \int_0^2 x(1-x)^2\,dx$$

$$= \frac{4}{3} - \int_0^2 x - 2x^2 + x^3\,dx$$

$$= \frac{4}{3} + \frac{2}{3} = 2$$