16.2 Line Integrals.

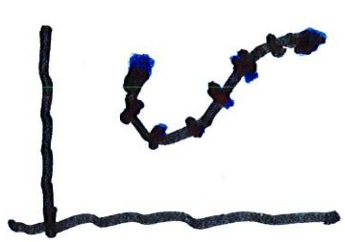
Suppose we are given a curve C

defined by $\{x(t), y(t)\}$ for $a \le t \le b$.

We want to define the integral

$\int_C f(x,y)\,ds$ of a function $f(x,y)$

that is defined for all (x,y) on C.



As usual, we partition the curve by defining points

$(x_i, y_i) = (x(t_i), y(t_i))$,

where $a < \dots < t_{i-1} < t_i < \dots t_n = b$.

The i-th segment has length approximately =

$$l_i = \left| \left\langle x'(t_i), y'(t_i) \right\rangle \right| \Delta t.$$

To define $\displaystyle\int_C f(x,y)\,ds$,

we multiply ℓ_i by $f(x_i, y_i)$

Thus, we obtain

$$\int_C f(x,y)\,ds \approx \sum_{i=1}^{n} f(x_i, y_i)\cdot$$

$$\overbrace{\left|\langle x'(t_i), y'(t_i)\rangle\right|}\,\Delta t$$

Letting $n \to \infty$, we get

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))$$

$$|\langle x'(t), y'(t)\rangle|\,dt$$

Let S = straight path from

$\langle 1,2\rangle$ to $\langle 4,8\rangle$. Then

define $\int_S x^2\,ds$

$\vec{r}(t) = \langle 1, 2\rangle + t\langle 3, 6\rangle, \quad 0 \le t \le 1.$

$\therefore\ x = 1 + 3t, \quad y = 2 + 6t$

$\langle x', y'\rangle = \langle 3, 6\rangle$

$|r'(t)| = |\langle 3, 6\rangle| = \sqrt{45}$

$$\int_0^1 x^2\, |r'(t)|\, dt$$

$$= \int_0^1 (1 + 6t + 9t^2)\sqrt{45}\, dt$$

$$= \sqrt{45}\,(t + 3t^2 + 3t^3)\Big|_0^1$$

$$= 7\sqrt{45}$$

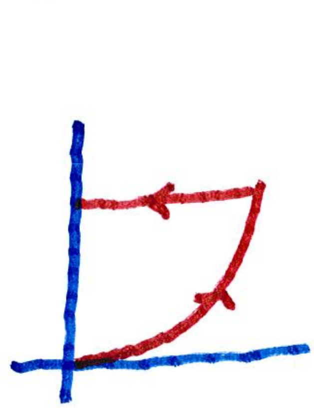
Ex. Compute $\displaystyle\int_{C_1} 2x\,ds + \int_{C_2} x^2,$

where C_1 = parabolic arc

$y = x^2$ from $(0,0)$ to $(1,1)$, and

where C_2 is the straight path

from $(1,1)$ to $(0,1)$



where C_2 is the straight path from $(1,1)$ to $(0,1)$.

Thus $\vec{r}_1(t) = \langle t, t^2 \rangle$

and $\vec{r}_2(t) = (1,1) - t(1,0)$

$= (1-t, 1)$

Note $|\vec{r}_1'(t)| = \sqrt{1+4t^2}$

$\rightarrow \int_{C_1} 2x\,ds = \int_0^1 2t\sqrt{1+4t^2}\,dt$

$$= \frac{1}{4}\int_0^1 \sqrt{1+4t^2}\cdot 8t\,dt$$

$$= \frac{1}{4}\,(1+4t^2)^{3/2}\cdot\frac{2}{3}\Big|_0^1$$

$$= \frac{1}{4}\,\frac{2}{3}\left\{5^{3/2} - 1^{3/2}\right\}$$

$|r'(t)| = 1$

$$\int_{C_2} x^2\,ds = \int_0^1 (1-t)^2\cdot 1\,dt$$

$$\frac{(1-t)^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$= \frac{2}{8} \int_0^1 \sqrt{1+4t^2} \;\; 8t\, dt$$

$$\frac{2}{3} \cdot \frac{1}{4} \left(1+4t^2\right)^{3/2} \cdot \Big|_0^1$$

$$= \frac{1}{6} \left(5^{3/2} - 1 \right)$$

$$\int_{C_2} x^2\, ds = \int_0^1 (1-t)^2 \cdot 1 \cdot dt$$

$|\vec{r}\,'(t)| = 1$

$$= \int_0^1 (t-1)^2\, dt = + \frac{(t-1)^3}{3} \Big|_0^1$$

$$- \frac{(0-1)^3}{3} = \frac{1}{3}.$$

$$\therefore \int_{C_1} 2x \, ds + \int_{C_2} x^2 \, ds$$

$$= \frac{1}{6}\left(5^{\frac{3}{2}} - 1\right) + \left(\frac{1}{3}\right)$$

We can describe the integral of a function along a curve by

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

dt

The value of the line integral does not depend on the parameterization, provided that the curve is traversed exactly once as t increases from a to b.

If C is a union of a finite number of curves (that are piecewise smooth.)

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds$$

Ex. Let C be the curve

satisfying $y = x^2$, $|x| \le 2$

Assume the density depends

on the curve, say $\rho = 4 - y$

$\vec{r}(t) = (t, t^2)$

$x = t$, $y = t^2$

$-2 \le t \le 2$

$-2 \le t \le 2$

mass $m = \displaystyle\int_{-2}^{2} (4 - y)\sqrt{(x')^2 + (y')^2}\, dt$

$y = y(t)$

$x' = 1$, $y' = 2t$

$$= \int_{-2}^{2} (4-t^2)\sqrt{1^2 + 4t^2}\ dt,$$

We can replace Δs_i by

$\Delta x_i = x_i - x_{i-1}$ or

$\Delta y_i = y_i - y_{i-1}$.

Then one obtains

$$\int_C f(x,y)\,dx = \int_a^b f(x(t), y(t))x'(t)\,dt$$

or $$\int_C f(x,y)\,dy = \int_a^b f(x(t), y(t))\,y'(t)\,dt$$

Line Integrals in Space

Suppose a curve C is given by

$x = x(t), \quad y = y(t), \quad z = z(t).$

Given a fcn. $f(x, y, z)$, we obtain

$$\int_C f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

This can be written more compactly

as $\displaystyle\int_a^b f(\vec{r}(t)) \, |r'(t)| \, dt$

Ex. Evaluate the line integral

$$\int_C y\,dx - 3x\,dy \,,$$

where $x = 2+3t$, $y = 1-4t$, $0 \le t \le 1$

and $x' = 3$, $y' = -4$

$$\therefore \int_C (1-4t)(3) - 3(2+3t)(-4) \quad 0 \le t \le 1$$

$$= \int_0^1 (17+24) + 36t \; dt$$

$$= 41 + 18 = \underline{\underline{59}}$$