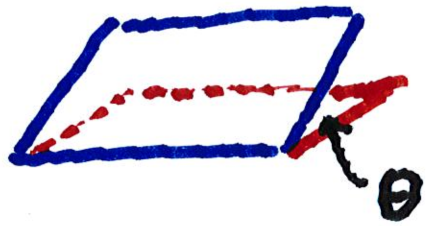
If 2 planes are not parallel, then they intersect in a straight line and the angle between them is defined as the acute angle between the normal vectors.




Ex. Find the angle between

$2x + y + z = 4$ and $x - 2y + z = 1$.

$\vec{n}_1 = \langle 2, 1, 1 \rangle$ $\qquad$ $\vec{n}_2 = \langle 1, -2, 1 \rangle$

$$\therefore \cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{2-2+3}{\sqrt{6}\,\sqrt{6}}$$

$$= \frac{3}{6} = \frac{1}{2} \qquad \therefore \theta = \frac{\pi}{3}$$

Ex. Express the line of intersection of the above using symmetric equations. Note that the above line lies in both planes $P_1$ and $P_2$.

$\therefore$ If $\vec{v}$ is the direction vector of the line $L$, then

$\vec{v}$ is $\perp$ to $\vec{n}_1$ and $\vec{v}$ is $\perp$ to $\vec{n}_2$.

$$\therefore \quad \vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 3\vec{i} - \vec{j} - 5\vec{k}.$$

We still need a point $P$ in $L$.

Set $z = 0$ $\rightarrow$ $x + y = 4$

$x - 2y = 1$

$\rightarrow$ $y + ($ $3y = 3$ $\rightarrow$ $y = 3/4$

and $x = 13/4$

$\therefore \left\langle \frac{13}{4}, \frac{3}{4}, 0 \right\rangle$ is in L.

$$\therefore \langle x, y, z \rangle = \left\langle \frac{13}{4}, \frac{3}{4}, 0 \right\rangle + t\langle 3, -1, 5 \rangle$$

Sym. Eq'ns are

$$\frac{x - \frac{13}{4}}{3} = \frac{y - \frac{3}{4}}{-1} = \frac{z}{-5}.$$

# 12.6 Surfaces in 3-dimensions

The cylinder is described by

$$x^2 + y^2 = r^2$$

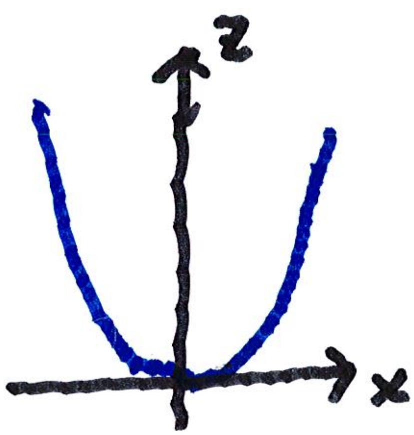
Note that the equation is independent of $z$.

Thus $z$ can assume any value.

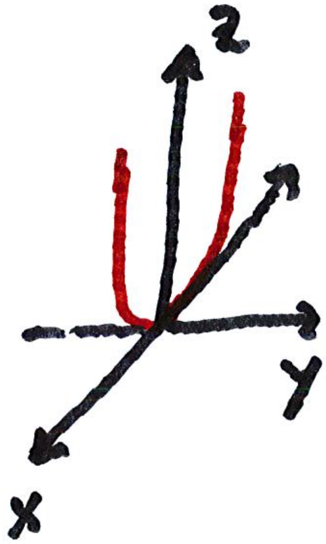
Ex. Sketch the surface

(ind. of y)

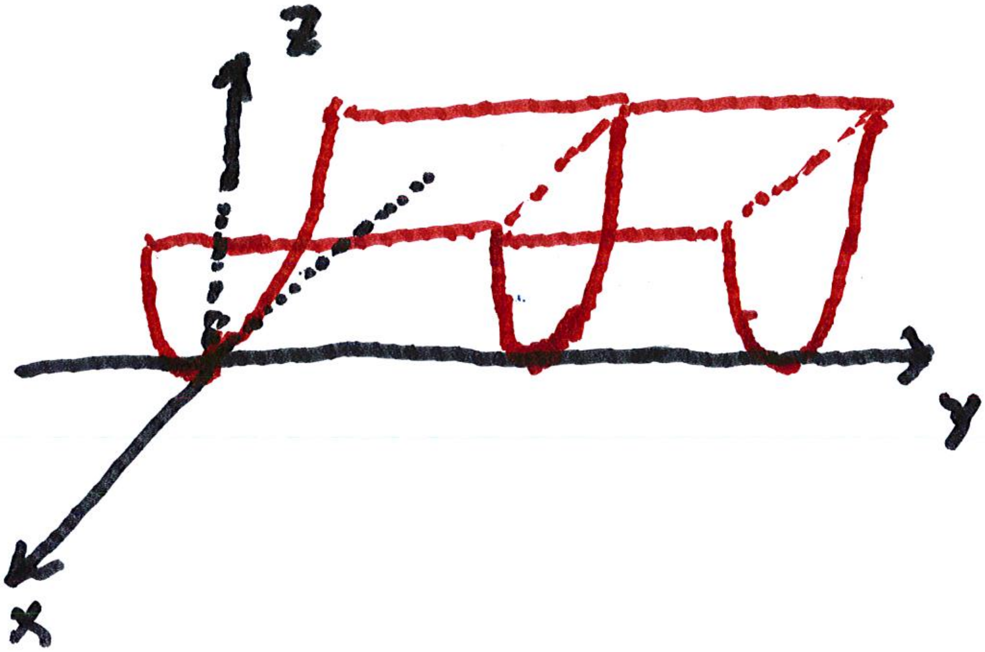
$z = x^2$ First draw the curve in the xz-plane.




Then draw curve in the xz-plane in 3 dim.

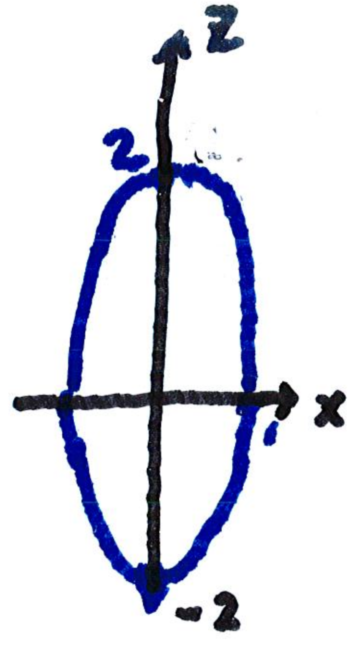



Then slide the curve in y-direction



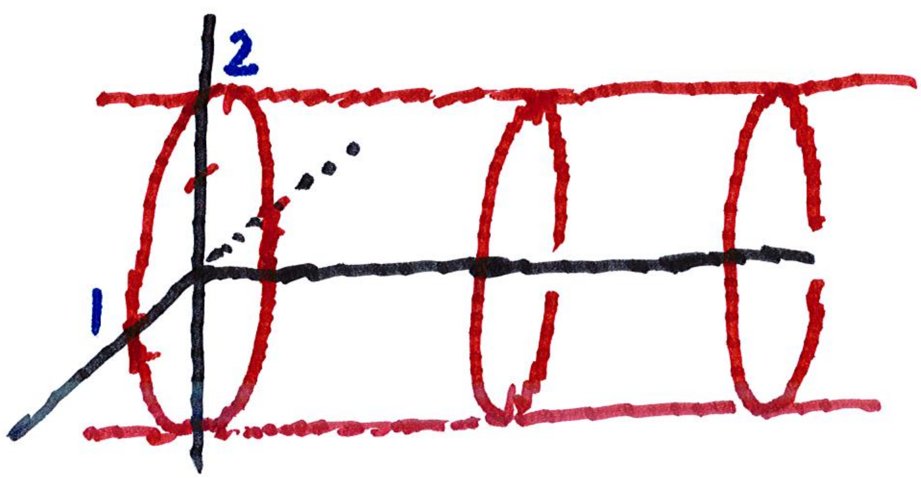

Ex Sketch the cylinder

$$x^2 + \frac{z^2}{4} = 1$$



The equation is independent of y.

Now slide the ellipse in the y-direction



# Quadric Surfaces.

This is the graph of second-degree equation satisfying

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

By making translations and rotations, this can be put in the form

I $\quad Ax^2 + By^2 + Cz^2 + J = 0$ or

II $\quad Ax^2 + By^2 + Iz = 0$

Ex. Suppose in type I that A, B, and C are all positive.

Ex. $\quad x^2 + \frac{y^2}{4} + \frac{z^2}{16} = 1$.

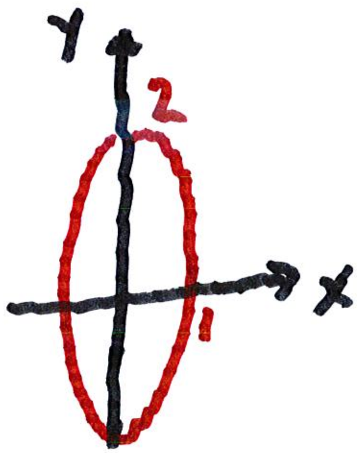
This is an ellipsoid.

We can write this as

$$x^2 + \frac{y^2}{4} = 1 - \frac{z^2}{16}$$

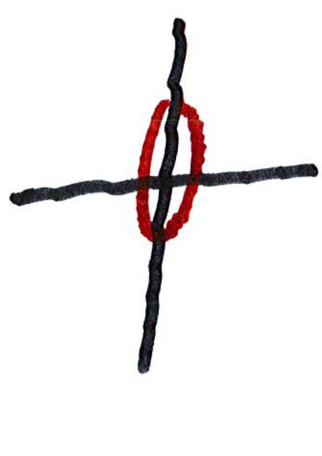
When $z = 0$, the "trace"

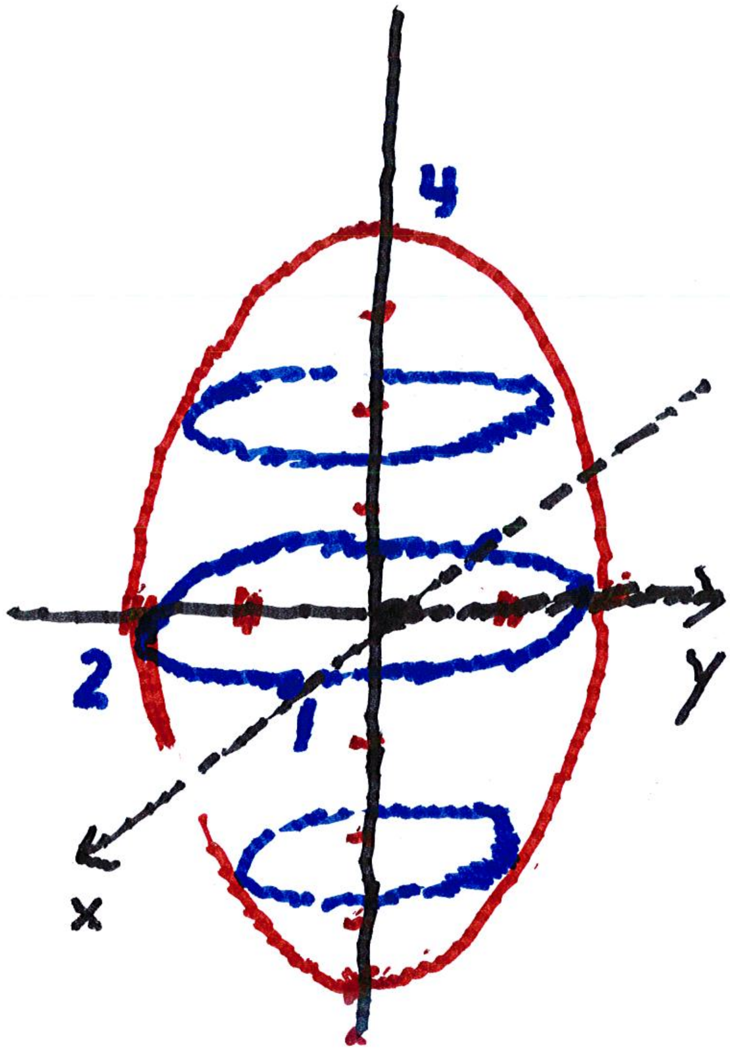
is $$\frac{x^2}{1} + \frac{y^2}{4} = 1$$




As $z$ increases

$1 - \frac{z^2}{16}$ decreases






The ellipses are traces for different values of z $(-4 \le z \le 4)$

The trace is the cross-section obtained by fixing one of the variables.

We often analyze a surface by finding various traces.

Ex. Now consider the elliptic paraboloid.

$$z = x^2 + \frac{y^2}{4}$$

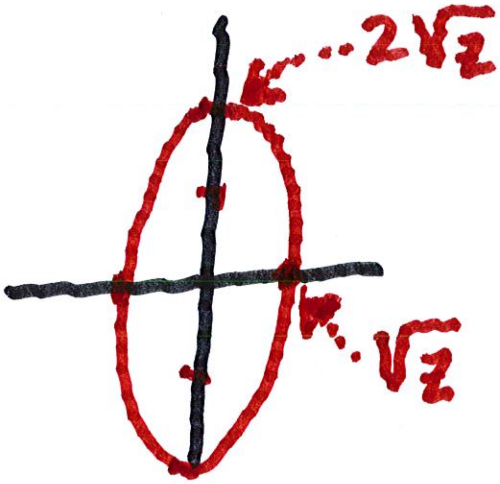
If $z = 0$, then $x = 0$, $y = 0$

If $z > 0$,

$$(\sqrt{z})^2 = x^2 + \frac{y^2}{4}$$

$$\rightarrow \quad 1 = \left(\frac{x}{\sqrt{z}}\right)^2 + \left(\frac{y}{2\sqrt{z}}\right)^2$$

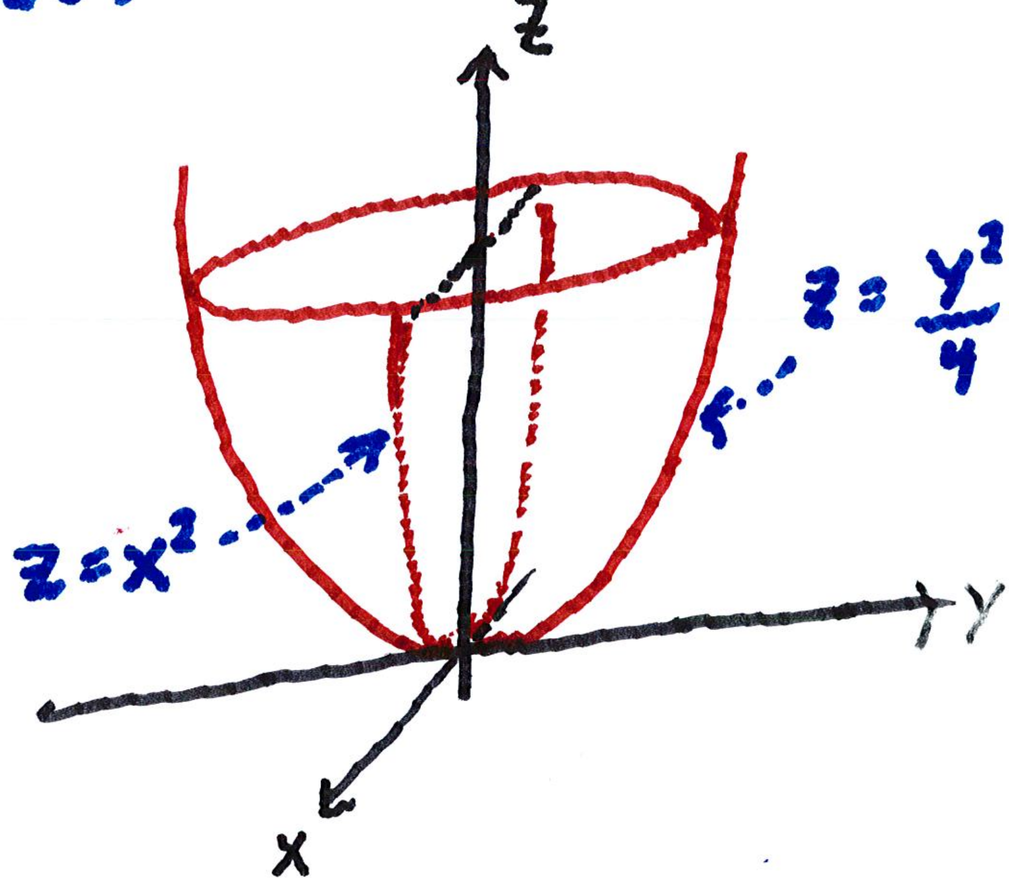
This is an ellipse




, which gets bigger as $\sqrt{z}$ increases.

If $y=0$, the trace is $z=x^2$, and if $x=0$, the trace is $z=\frac{y^2}{4}$

We get:




Ex. The hyperbolic paraboloid

is given by $z = y^2 - x^2$;

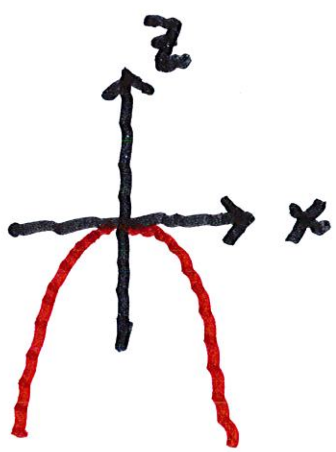
For fixed y, $z = y^2 - x^2$

(if $y = 0$) $z = -x^2 + y^2$

Ex. The hyperbolic paraboloid is given by

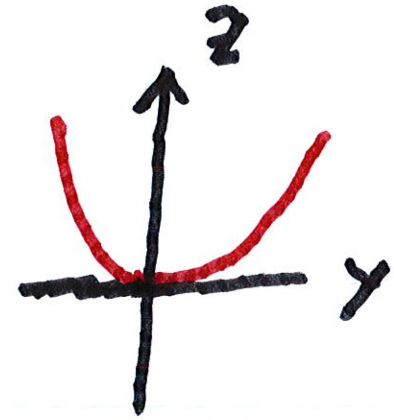
$z = y^2 - x^2$

For $y = 0$ the surface is given by $z = -x^2$

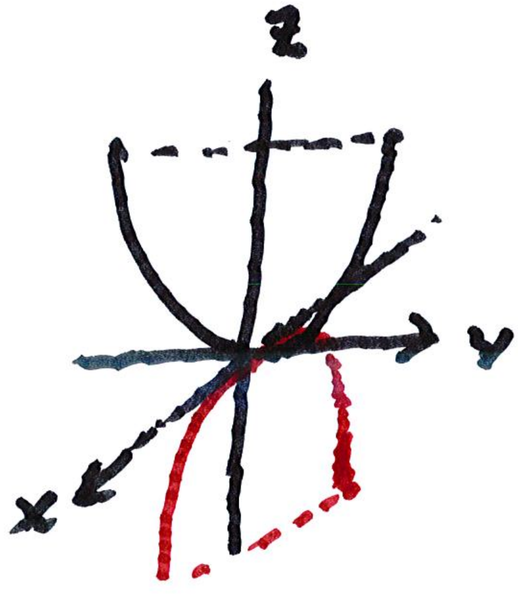



and if $x = 0$,

and the trace of $x = 0$ is $z = y^2$

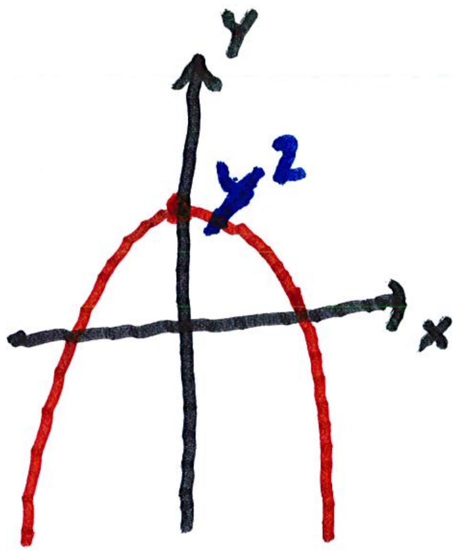



In 3 dimensions:



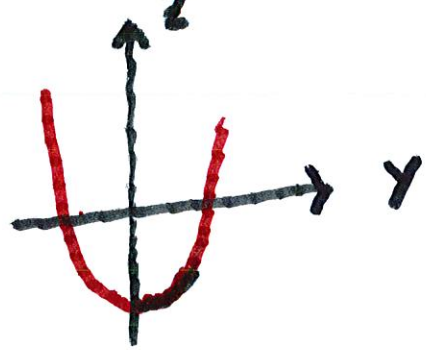

More generally, for fixed $y$,

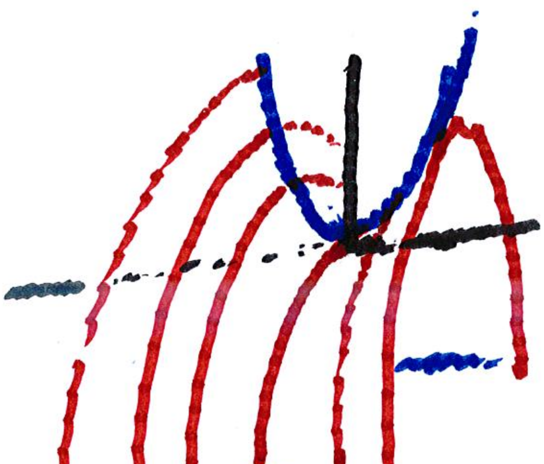
the curve is $z = y^2 - x^2$




The general surface is:

For fixed $x$, $z = y^2 - x^2$

The origin $(0,0,0)$ is a saddle point

Ex. Now consider the

equation

$$x^2 + y^2 - z^2 = \underbrace{-4}_{<0}$$

or $x^2 + y^2 = z^2 - 4$

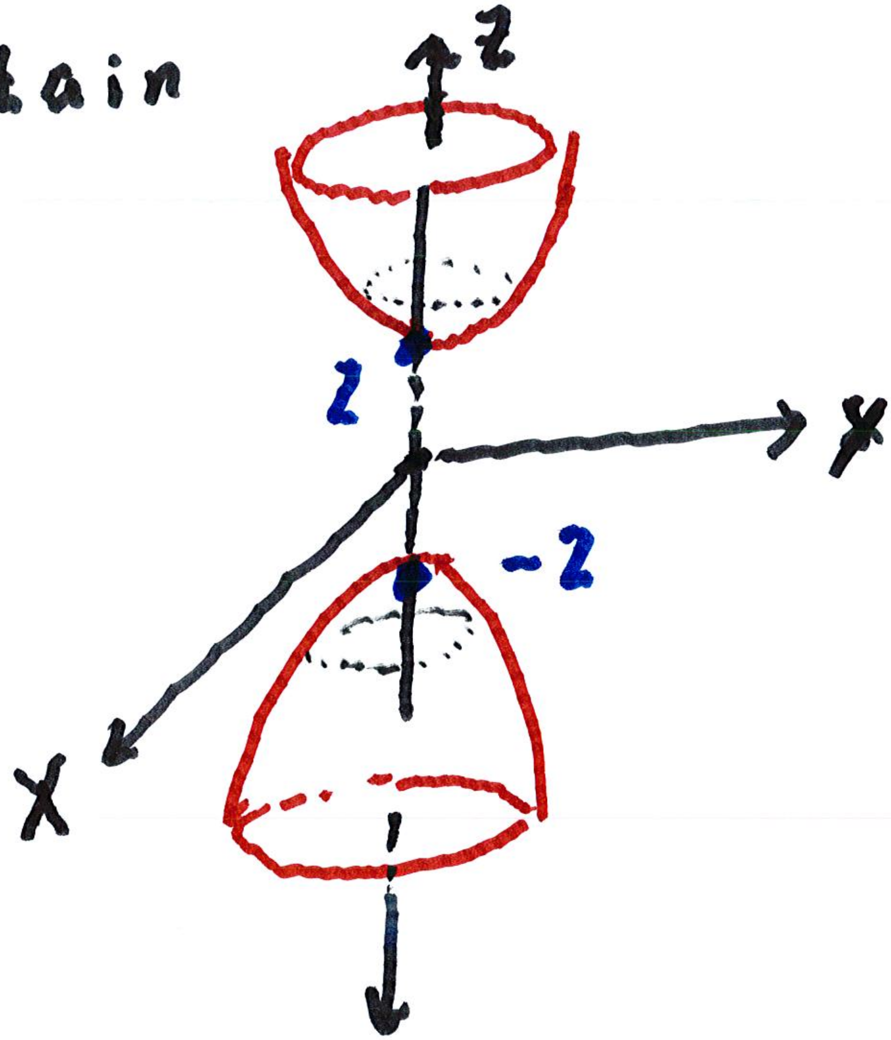
There is no solution

if $z^2 < 4$. If $z^2 \geq 4$

we have a circle of

radius of $\sqrt{z^2-4}$.

We obtain




This is a "hyperboloid

of 2 sheets

Now suppose that

2 of the coefficients

A, B, C are positive

and the other is negative.

Ex. $x^2 + y^2 - z^2 = 4 \leftarrow > 0$

or $x^2 + y^2 = 4 + z^2$

For fixed z, the

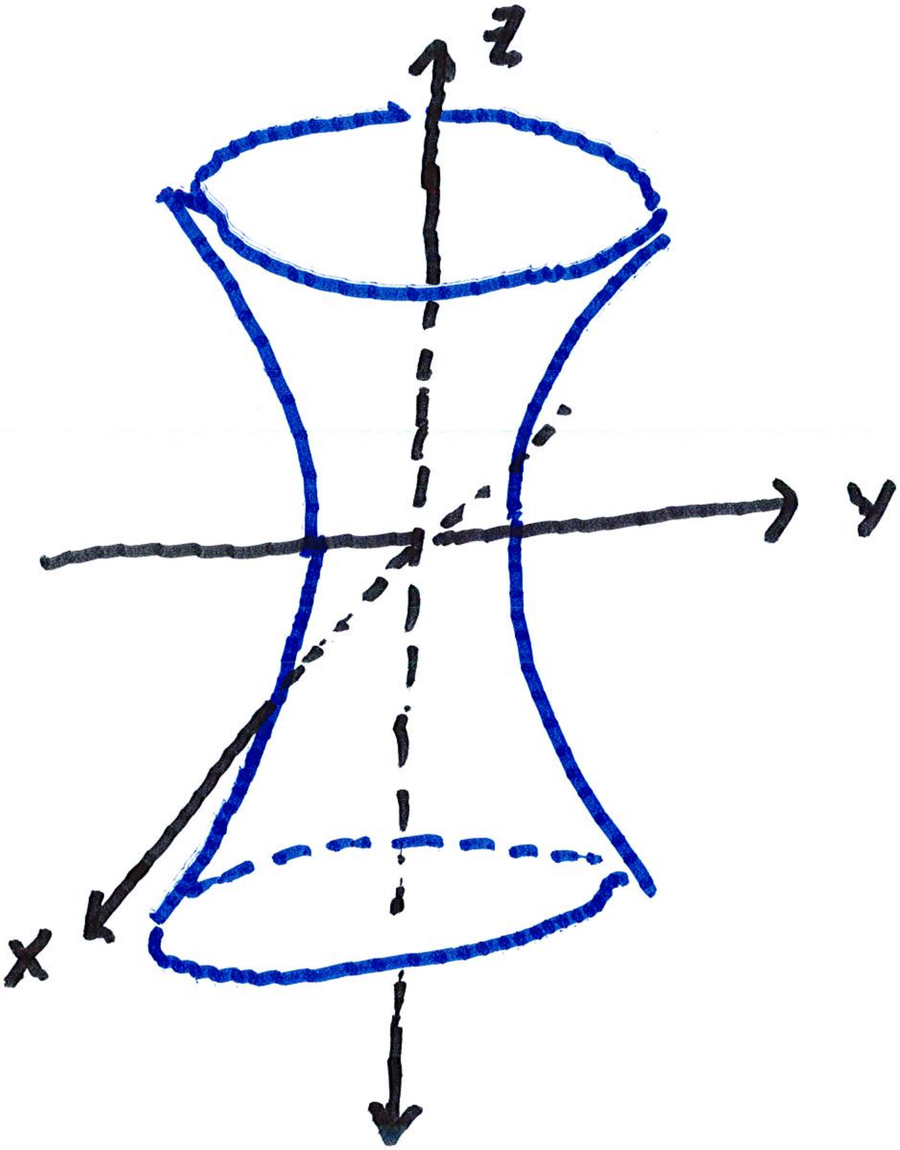
trace is

$$x^2 + y^2 = \left(\sqrt{4+z^2}\right)^2$$

This is a circle of

radius $\sqrt{4+z^2} \to \infty$ as

$z, -z \to \infty$




This is called by a

"hyperboloid of 1 sheet"

Ex. Now suppose that

$$x^2 + y^2 - z^2 = 0$$

or $x^2 + y^2 = z^2 = |z|^2$

This is a circle of

radius $|z|$. We obtain

a <u>cone</u>