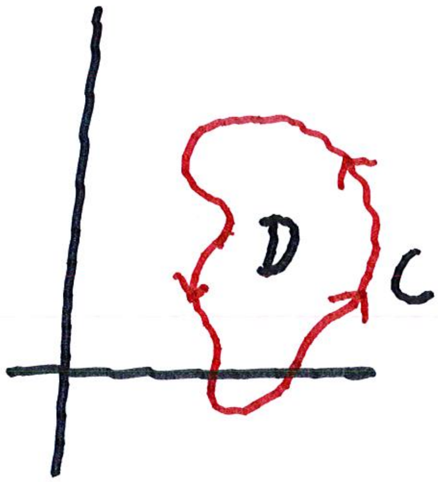
16.4 Green's Theorem

Let C be the boundary of a region D. We adopt the convention the curve C has a positive orientation if C has the counterclockwise orientation.

Thus if C is parameterized by $\vec{r}(t)$, $a \le t \le b$, then the domain D should be on the left side



Green's Thm. Let C be positively oriented, and suppose also that the boundary of D is parameterized by a single closed curve C.

If P and Q have continuous partial derivatives, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Sometimes you'll see the notation

$\oint_C P\,dx + Q\,dy$ to emphasize that C is parameterized counter-clockwise.

In one variable, the Fund. Thm of Calculus states that

$$\int_a^b \underbrace{F'(x)\,dx}_{\text{On the interior of } [a,b]} = \underbrace{F(b)-F(a)}_{\text{On the boundary of } [a,b]}$$

Why is Green's Thm. true?

We assume that D is of type I. (Maybe later)

Ex. Use Green's Thm. to

evaluate $\displaystyle\int_C 2xy\,dx + x^2\,dy$,

where C is the triangle with

vertices at $(0,0)$, $(1,1)$, $(1,3)$

$P(x,y) = 2xy +$

$Q(x,y) = x^2 - x$

$\dfrac{\partial Q}{\partial x} = 2x-1 \qquad \dfrac{\partial P}{\partial y} = 2x$

$$\therefore \int_C P\,dx + Q\,dy$$

$$= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

$$= \iint_D \Big((2x-1) - (2x)\Big)\,dA$$

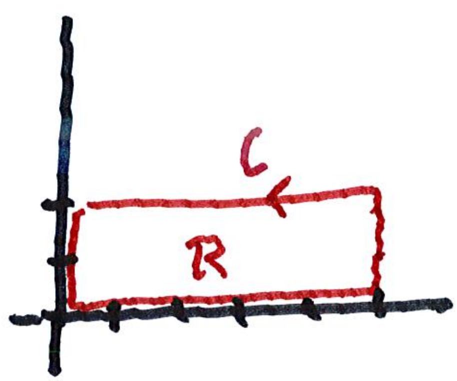
$$= -\iint_D = -A = (1 \cdot 2 \cdot \tfrac{1}{2}) = -1$$

Ex. Compute $\int_C \cos y \, dx + x^2 \sin y \, dy$,

where C = rectangle with

vertices at $(0,0)$, $(5,0)$, $(5,2)$

and $(0,2)$.



$P(x,y) = \cos y$

$Q(x,y) = x^2 \sin y$

$$\int_C = \iint_R (2x\sin y) + \sin y \, dA$$

$$= \int_{0}^{5} \int_{0}^{2} (2x+1) \sin y \, dy \, dx$$

$x \quad y$

$$= \int_{0}^{5} (2x+1)(-\cos y)\Big|_{0}^{2}$$

$$= \int_{0}^{5} (2x+1)(-\cos 2 + 1)\, dx$$

$$= (1-\cos 2)(x^2+x)\Big|_{0}^{5}$$

$$= (1-\cos 2)\cdot(30)$$

Ex. Use Green's Thm. to evaluate

$$\int_C \vec{F} \cdot d\vec{r}, \quad \text{where } \vec{F} = \langle y - \cos y, \ x \sin y \rangle$$

and C = circle

$$(x-3)^2 + (y+4)^2 = 4.$$

cent. = $(3, -4)$

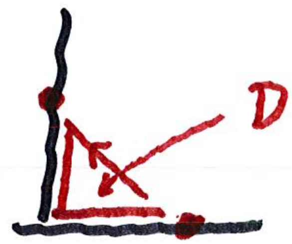
$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA =$$

$$= \iint_D \sin y - (1 + \sin y) \, dA$$

$$= -\iint_D 1 \, dA = -\pi \cdot 2^2 = -4\pi$$

Ex. Use Green's Thm. to find the work done by the force $\vec{F} = x(x+y)\vec{i} + xy^2\vec{j}$ in moving a particle from the origin along the x-axis to (1,0), then along the line segment to (0,1) and then back along the y-axis

to the origin



$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (y^2 - x)$$

$$\therefore \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \iint_D y^2 - x \, dA$$

$$= \int_0^1 \int_0^{1-x} (y^2 - x) \, dA$$

$$= \int_0^1 \left(\frac{y^3}{3} - xy\right)\Big|_0^{1-x}$$

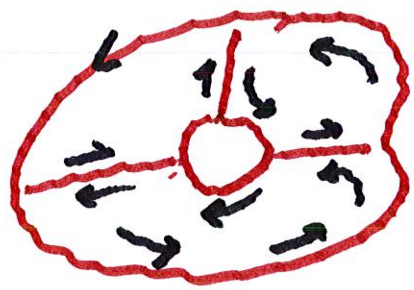
$$\int_0^1 \frac{(1-x)^3}{3} - x(1-x) \quad dx$$

$$= \int_0^1 \frac{1}{3} - x + x^2 - \frac{x^3}{3} - x + x^2 \quad dx$$

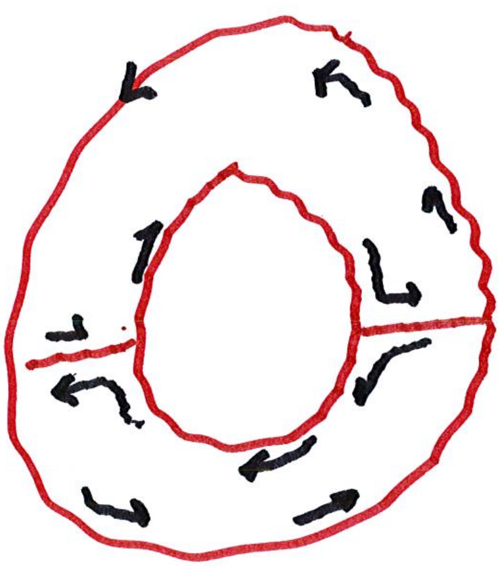
$$= \int_0^1 \frac{1}{3} - 2x + 2x^2 - \frac{x^3}{3} \quad dx$$

$$= \quad = \frac{1}{3} - 1 + \frac{2}{3} - \frac{1}{12} = -\frac{1}{12}$$

What about more complicated regions



Note that the inside is parameterized clockwise.



We can use Green's Thm.

to compute the area of a

domain D

There are 3 ways:

1. Set $P = 0$, $Q = x$

$\therefore Q_x - P_y = 1 - 0 = 1$

2. Set $P = -y$, $Q = 0$

$\therefore Q_x - P_y = 0 - (-1) = 1$

or

3. $Q = \frac{1}{2} x$, $P = -\frac{1}{2} y$

$\therefore Q_x - P_y = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$

Ex. Find area of ellipse. (Use 3)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$A = \frac{1}{2} \int_C x\,dy - y\,dx = \frac{1}{2} \iint_D 1+1\,dA = A$$

Set $x = a\cos t$ $y = b\sin t$

$$A = \frac{1}{2} \int_C x\,dy - y\,dx$$

$$= \frac{1}{2} \int_0^{2\pi} a\cos t \cdot b\cos t - b\sin t \cdot (-a\sin t)\,dt$$

$$= \frac{1}{2} \int_0^{2\pi} ab\,(\cos^2 t + \sin^2 t)\,dt$$

$$= \pi ab$$

Ex. 10 Let D be the region

between the circles $x^2+y^2=4$

and $x^2+y^2=9$

C = boundary of D 

$$\int_C (1-y^3)\,dx + (x^3+e^{y^2})\,dy$$

$$= \iint_D (3x^2+0)-(0-3y^2)\,dA$$

$$= 3\iint_D x^2+y^2\,dA$$

$$= 3\int_0^{2\pi}\int_2^3 r^2 \cdot r \, dr \, d\theta$$

$$= 6\pi \left. \frac{r^4}{4} \right|_2^3$$

$$= \frac{3\pi}{2}(81 - 16)$$