Math 341. An Introduction to Real Analysis

My name is David Catlin

Office is 744, Math. Bldg.

My email address is

catlind@purdue.edu.

My office hours are

Mon. 9:30
Wed. 10:30 Thur. 9:30

The text is:

An Introduction to

Real Analysis, 4th Edition

by Bartle and Sherbert.

The lectures and homework

exercises will be posted

online at math.purdue.edu/~catlin

the course homepage.

In this course we will give a rigorous and detailed study of the ideas and techniques of calculus of calculus of one variable, including

1. Set Theory
2. Real Numbers

3. Sequences and Series

4. Limits

5. Continuous Functions

6. Differentiation

7. Riemann Integral

8. Sequences and Series of Functions

9. Taylor Series

1.1 Sets and Functions

If x is in a set A, we write

$x \in A$

We also say x is a member of A or that x belongs to A. If x is <u>not in</u> A, we write $x \notin A$.

If every element of a set belongs to a set B, we say A is a subset of B, and

$A \subseteq B$ or $B \supseteq A$.

Some common sets of numbers are :

$\mathbb{N} = \{1, 2, 3, \ldots\}$ natural numbers

$\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$ integers

$\mathbb{Q} = \{m/n : m, n \in \mathbb{Z},\ n = 0\}$

rational numbers

$\mathbb{R}$ = set of real numbers

Sometimes a set A is obtained by specifying a property that determines the elements of A.

Ex. We say n is an even integer if there is an integer k, so that $n = 2k$.

$$E = \{ n \in \mathbb{Z} : n = 2k, \text{ for any } k \in \mathbb{Z} \}$$

or

$$E = \{ 2k : k \in \mathbb{Z} \}$$

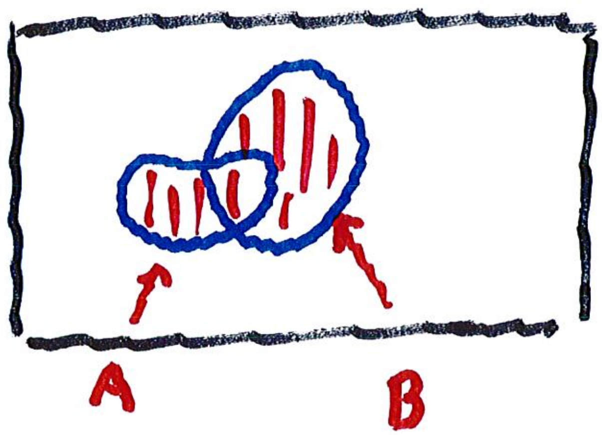
Ex. Let $I = \{ x \in \mathbb{Q} : 0 < x \text{ and } x^2 < 2 \}$

Set Operations

Def (a). The union of sets A and B is

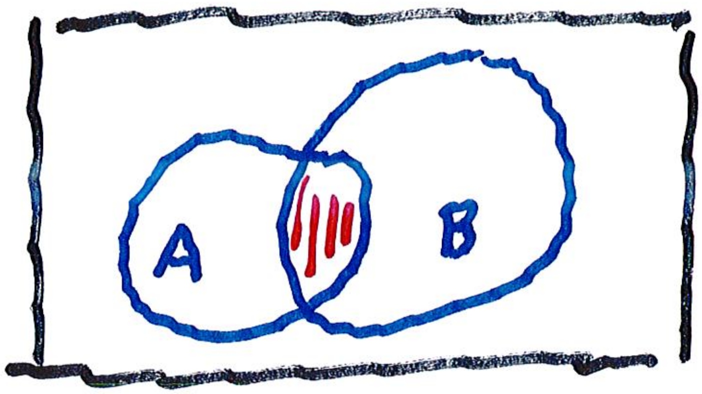
$$A \cup B = \{ x ; x \in A \text{ or } x \in B \}$$

(x can be in both)



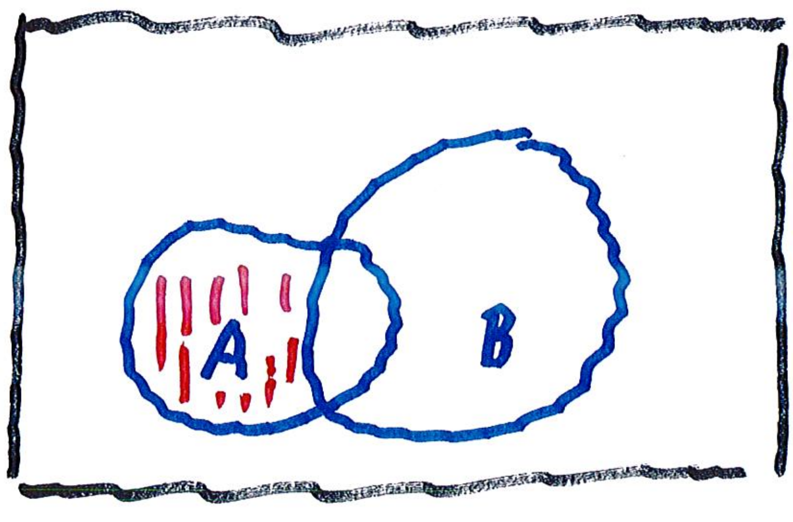
(b) The intersection of the sets A and B is the set

$$A \cap B = \{x \,;\, x \in A \text{ and } x \in B\}$$



(c) The complement of B relative to A is the set

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$



The set with no elements is the empty set, written as $\emptyset$

Two sets A and B are said to be disjoint if there is no element in both A and B.

A and B are disjoint if $A \cap B = \emptyset$

Here's a way to show two sets are equal:

De Morgan laws for three sets.

Thm. If A, B and C are sets, then

(a) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

(b) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Pf. of (b). Assume first that $x \in A \setminus (B \cap C)$. Then $\underline{x \in A}$ and $x \notin (B \cap C)$. Note that

$x \notin (B \cap C) \Rightarrow \underline{x \notin B}$ or $\underline{x \notin C}$.

Hence, $x \in A \setminus B$ or $x \in A \setminus C$, which implies that

$x \in (A \setminus B) \cup (A \setminus C)$.

Thus,

$A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$.

Now assume that

$x \in (A \setminus B) \cup (A \setminus C)$.

Then either

$x \in A$ and $x \notin B$, or

$x \in A$ and $x \notin C$.

Note that

$x \notin B$ or $x \notin C \Rightarrow x \notin (B \cap C)$.

Hence $x \in A \setminus (B \cap C)$.

This implies that

$$(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C).$$

which shows that (b) holds.

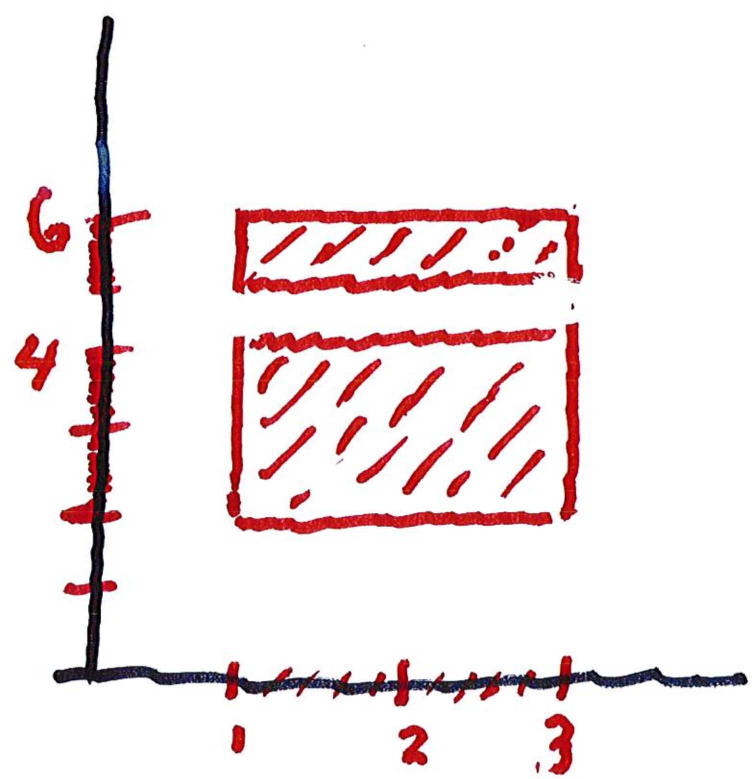
Functions

Def'n. If A and B are nonempty, then the Cartesian product $A \times B$ is

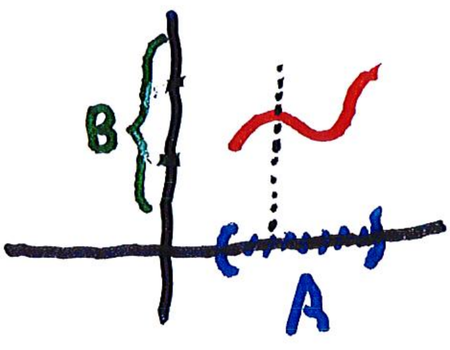
$$A \times B = \left\{ (a, b) : a \in A,\ b \in B \right\}$$

If $A = \left\{ x : 1 \le x \le 3 \right\}$

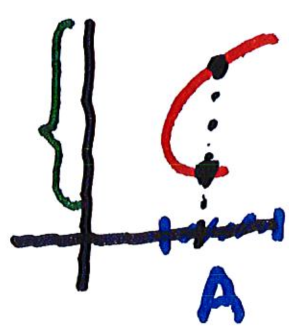
and $B = \left\{ y : 2 \le y \le 4 \text{ or } 5 \le y \le 6 \right\}$



A function f from A to B is a set f of ordered pairs in $A \times B$ such that for each a in A, there is unique b in B such that $(a, b) \in f$

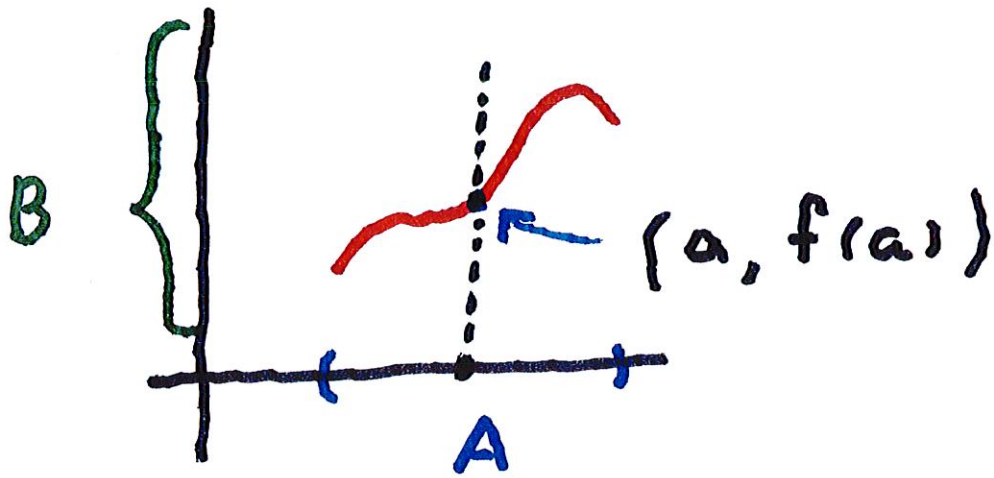


is a fcn.



not a fcn.

If $(a,b) \in f$, we often write $f(a) = b$



We write Domain $= D(f) = A$

Also $R(f) = \{ f(a) : a \in A \}$

Composition of Fcn's.

If A, B, and C are maps,

and $f: A \to B$ and $g: B \to C$

then the composition of f and g is

$$(g \circ f)(x) = g(f(x))$$

for all x in A

Ex. Suppose $f(x) = x^4 - 1$ for x in $(-\infty, \infty)$

and $g(x) = \sqrt{x}$, for
$0 \le x < \infty$,

then we cannot form

$$(g \circ f)(x) = \sqrt{x^4 - 1}.$$

The problem is $x^4 - 1 < 0$
if $-1 < x < 1$,

because $\sqrt{x^4-1}$ only makes sense
if $x^4 - 1 \ge 0$, i.e., if $|x| \ge 1$.

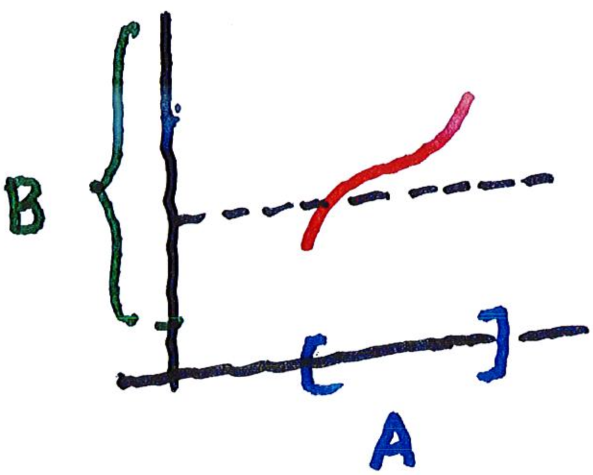
Then we modify f by defining $f(x) = x^4 - 1$ for $|x| \geq 1$.

Definition. A function $f : A \to B$ is injective, if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. {f is 1-to-1}

Equivalently, if whenever $f(x_1) = f(x_2)$, then $x_1 \neq x_2$.

Also $f: A \to B$ is surjective if whenever $y \in B$, then there is an x in A so $f(x) = y$

(f is onto)



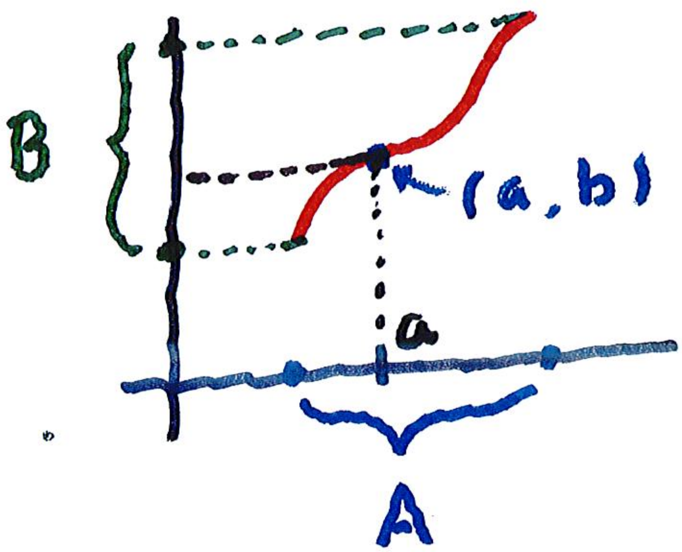
f is 1-to-1 but not onto

f is onto but not 1-to-1

We say f is bijective

if f is both <u>injective</u>

and <u>surjective</u>.



$f(a) = b$

$g(b) = a.$

Theorem. Suppose $f: A \longrightarrow B$

is bijective, (i.e., both
onto and
1-to-1)

Then there is a bijection

$g: B \longrightarrow A$ that satisfies

(i) $g(f(a)) = a$ for all
a in A

(ii) $f(g(b)) = b$ for all
b in B

First we define $g(b)$ for any b in B

Since f is onto, there is one element a in A so that

(1) $f(a) = b$. Moreover there is only one such a.

For if $\tilde{a} \in A$ with $f(\tilde{a}) = b$, and if $\tilde{a} \neq a$, then this would mean $f(a) = f(\tilde{a})$, which contradicts the fact that f is 1-to-1. Hence we define $g(b) = a$. (2)

Now we show that (ii) holds. Apply f to both sides of (2)

$\Rightarrow$ $f(g(b)) = f(a) = b$

by (1).

Since b is arbitrary, this proves (ii)

Now let a be in A.

Then $b = f(a)$, and as we saw above, $g(b) = a$. (3)

If we apply g to both sides of $b = f(a)$, we get

$$g(b) = g(f(a)).$$

so according to (3)

$$a = g(f(a)),$$ for all a in A.

This proves (i).

Finally, note that (i)

proves that for any a in A,

g maps $f(a)$ to a. $\therefore g$ is onto

Also, if $g(b_1) = g(b_2)$

then (a) shows that

$$f(g(b_1)) = f(g(b_2)),$$

which by (i) implies that

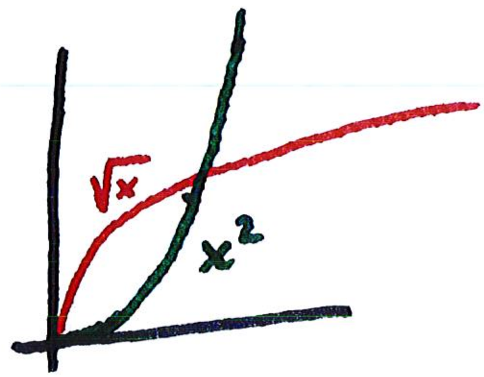
$b_1 = b_2$. Hence g is 1-to-1.

and so, g is bijective. ✓

Ex. Let $f(x) = x^2$. The $(0 \le x < \infty)$

inverse

of f is $\sqrt{x}$.



$x^2 = 3$

Apply $\sqrt{}$: $\sqrt{x^2} = \sqrt{3}$

or $x = \sqrt{3}$.

Ex. $\sin x$ maps $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to $[-1,1]$

$\sin^{-1}$ maps $[-1, 1]$ to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Suppose $\sin x = .42$

Apply $\sin^{-1}$:

$$\sin^{-1}(\sin x) = \sin^{-1}(.42)$$

$$\rightarrow \quad x = \sin^{-1}(.42)$$

Ex. Let $A = \{x \in \mathbb{R} : x \neq -1\}$

and let $f(x) = \dfrac{2x-1}{x+1}$.

Show that f is injective.

Suppose $f(x_1) = f(x_2)$

$$\frac{2x_1 - 1}{x_1 + 1} = \frac{2x_2 - 1}{x_2 + 1}$$

$$(2x_1 - 1)(x_2 + 1) = (2x_2 - 1)(x_1 + 1)$$

$$2x_1 - x_2 = -x_1 + 2x_2$$

$$\rightarrow \quad 3x_1 = 3x_2$$

or $x_1 = x_2$. ✓

Now find the range of f.

Find all y, such that

$$y = \frac{2x-1}{x+1} \rightarrow yx + y = 2x - 1$$

Solve for x: $(y-2)x = -y-1$

$$\rightarrow \quad x = \frac{y+1}{2-y}$$

This can be solved only

if $y \neq 2$, $R(f) = \{ y \in \mathbb{R} : y \neq 2 \}$