Definition. We say a sequence (x_n) is increasing if

$$x_n \le x_{n+1}, \qquad \text{all } n = 1, 2, \ldots$$

That is

$$x_1 \le x_2 \le \ldots x_n \le x_{n+1} \le \ldots$$

We say (y_n) is decreasing if

$$y_n \ge y_{n+1}, \qquad n = 1, 2, \ldots$$

That is

$$y_1 \ge y_2 \ge \ldots \ge y_n \ge y_{n+1} \ge \ldots$$

If (x_n) is increasing or decreasing, we say (x_n) is monotone.

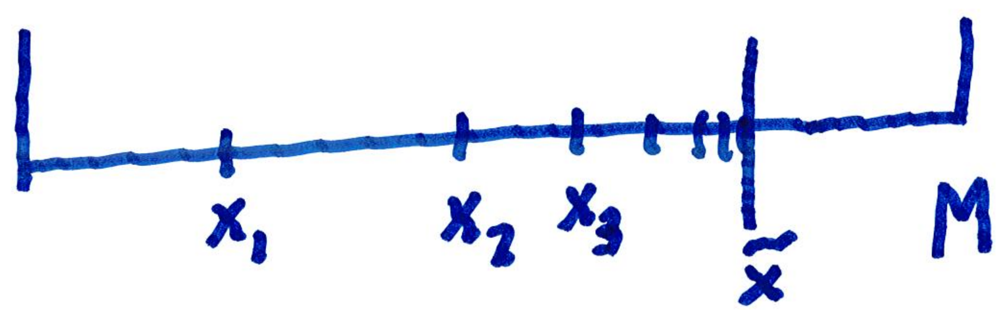
Monotone Convergence Thm.

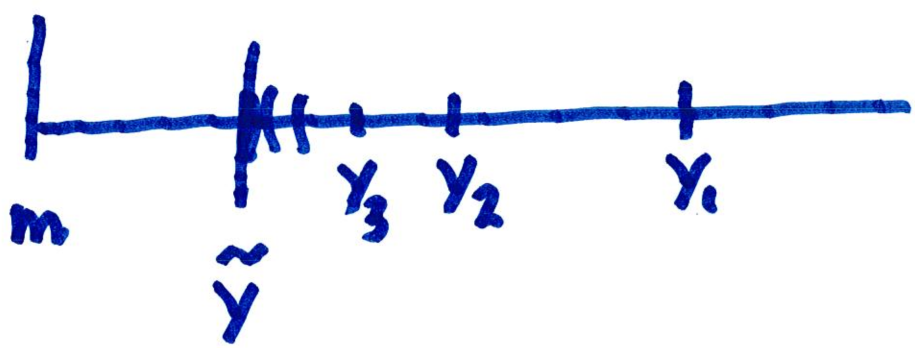
If (x_n) is a bounded monotone sequence, then it converges. In fact, if (x_n) is increasing and bounded, then

$$\lim(x_n) = \tilde{x} = \sup\{x_n : n \in \mathbb{N}\}$$

Also, if (y_n) is decreasing and bounded, then

$$\lim(y_n) = \tilde{y} = \inf\{y_n : n \in \mathbb{N}\}$$





Proof. Since $x_n \leq M$ for all $n \in N$, we define

$$\tilde{x} = \sup\{x_n : n \in N\}.$$

For any $\varepsilon > 0$, $\tilde{x} - \varepsilon$ is not an upper bound. It follows that there is a $K \in N$, such that $\tilde{x} - \varepsilon < x_K \leq \tilde{x}$.

Since (x_n) is increasing,

if $n \geq K$, then

$$\tilde{x} - \varepsilon < x_K \leq x_n \leq \tilde{x} < \tilde{x} + \varepsilon,$$

where the inequality

$x_n \leq \tilde{x}$ comes from the fact

that $\tilde{x}$ is an upper bound

of $\{x_n : n \in N\}$

It follows that $|x_n - \tilde{x}| < \varepsilon$

if $n \geq K$. Hence $\lim(x_n) = \tilde{x}$.

In the case of (y_n),

$y_n \geq m$ for all n, which

implies that there is a

number $\tilde{y} = \inf\{y_n : n \in N\}$

For any $\varepsilon > 0$, there is a K'

so that $\tilde{y} \leq y_K < \tilde{y} + \varepsilon$.

Since (y_n) is decreasing,

we obtain that if $n \geq K'$, then

$$\tilde{y} + \varepsilon > y_{K'} \geq y_n \geq \tilde{y} > \tilde{y} - \varepsilon,$$

or that

$$\tilde{y} - \varepsilon < y_n < \tilde{y} + \varepsilon, \text{ for } n \geq K'$$

It follows that $\lim(y_n) = \tilde{y}$, which proves the theorem.

We now use the Least Upper Bound Property to evaluate the lim of some sequences.

Ex. Define $y_{n+1} = \frac{2}{5} y_n + 1$,

with $y_1 = 1$.

Assume first

that $\lim y_n = y$. Then we have

$$y = \frac{2}{5} y + 1 \rightarrow \frac{3}{5} y = 1$$

$$\rightarrow y = \frac{5}{3}$$

Use induction to show that

if $1 \le y_n \le 4$, then y_{n+1} also satisfies $1 \leq y_{n+1} \leq 4$.

In fact, if $y_n \geq 1$, then

$$y_{n+1} = \frac{2}{5} y_n + 1 \geq \frac{2}{5} \cdot 1 + 1 > 1$$

Similarly, if $y_n \leq 4$, then

$$y_{n+1} = \frac{2}{5} y_n + 1 \leq \frac{8}{5} + 1 \leq \frac{13}{5}.$$

Now we show that $\underline{y_{n+1} > y_n}$

This is obvious when $n = 1$.

Now assume that $y_{n+1} > y_n$.

Then $\frac{2}{5} y_{n+1} > \frac{2}{5} y_n$,

which gives

$$\frac{2}{5} y_{n+1} + 1 > \frac{2}{5} y_n + 1$$

or $\underline{y_{n+2} > y_{n+1}}$.

Since $y_n \le 4$ for all $n \in \mathbb{N}$,

and since (y_n) is increasing,

we conclude that there

is a $y \in [1, 4]$ such that

$\lim(y_n) = y$ and $\lim(y_{n+1}) = y$.

This implies that

$$y = \frac{2}{5}y + 1 \to y = \frac{5}{3}.$$

Ex. Study the convergence of

$$y_n = \left\{ \frac{1}{n+1} + \cdots + \frac{1}{2n} \right\}.$$

Note that

$$y_{n+1} = -\frac{1}{n+1} + \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} \right\}$$

$$\to y_{n+1} = y_n + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= y_n + \frac{1}{(2n+1)(2n+2)}$$

$$\therefore y_{n+1} > y_n$$

Note also that

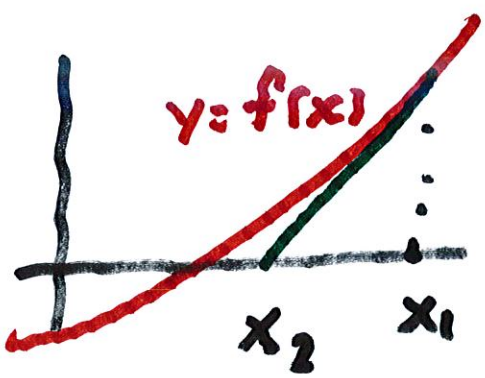
$$y_n \leq n \cdot \frac{1}{n+1} \leq 1 \text{ for all } n.$$

Hence the Monotone Convergence Theorem implies that

$$y_n \to y \leq 1 \text{ as } n \to \infty.$$

Approximating Roots by using Newton's Method.

Idea. To find $\tilde{x}$ so that $f(\tilde{x}) = 0$, suppose that $f(x_1)$ is small.



where the approximating line is tangent to $y = f(x)$ at $(x_1, f(x_1))$

$$y - f(x_n) = f'(x_n)(x - x_n)$$

If the line passes through

$y = 0$, we get

$$\frac{-f(x_n)}{f'(x_n)} = x - x_n$$

Solving for x,

$$x = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\rightarrow \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

If we use $f(x) = x^2 - a$,

then $$x_{n+1} = x_n - \frac{(x_n^2 - a)}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + a}{2x_n}$$

To approximate $\sqrt{a}$, we have

$$\therefore x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(known in Mesopotamia before 1500 B.C.)

Calculation of Square Roots.

Let $a > 0$. We construct

(s_n) that converges to $\sqrt{a}$.

(Known in Mesopotamia

before 1500 B.C.)

Let $s_1 > 0$ be arbitrary and

define $s_{n+1} = \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$

Note first that $s_n > 0$ for

all $n = 1, 2, \ldots$

We first show that

$$s_n^2 \geq a \quad , \qquad \text{all } n = 2, \ldots$$

Since s_n satisfies the quadratic equation

$$2 s_n s_{n+1} = s_n^2 + a.$$

or

$$(s_n - s_{n+1})^2 = s_{n+1}^2 - a.$$

Since the root s_n is real,

it follows that $s_{n+1}^2 - a \geq 0$

for all $n \geq 1$.

To see that (s_n) is ultimately

decreasing, we note that

for $n \geq 1$,

$$s_n - s_{n+1} = s_n - \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$$

$$= \frac{1}{2}\,\frac{(s_n^2 - a)}{s_n} \geq 0 .$$

Hence, $s_{n+1} \leq s_n$, for all $n \geq 2$.

The Monotone Convergence Thm. implies that $\lim(s_n)$ exists. It follows that

$$s = \frac{1}{2}\left(s + \frac{a}{s}\right)$$

and so, $s = \frac{a}{s} \rightarrow s^2 = a$