3.4. Subsequences

Let $X = (x_n)$ be a sequence

and let

$n_1 < n_2 < \dots < n_k < \dots$

be a strictly increasing

sequence of integers in $\mathbb{N}$.

Then the sequence

$X' = (x_{n_k})$ given by

$(x_{n_1}, x_{n_2}, \dots)$

is called a <u>subsequence</u>

of X.

Ex. $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\right)$

is a subsequence of

$\left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right) = X$

corresponding to $n_k = 2k$.

But $\left(\frac{1}{2}, \frac{1}{1}, \frac{1}{4}, \frac{1}{3}, \dots\right)$

is not a subsequence of X.

The following theorem is fundamental to the theory of calculus.

Bolzano-Weierstrass Thm.

A bounded sequence of real numbers has a convergent subsequence.

Pf. Since $\{x_n : n \in \mathbb{N}\}$

is bounded, this set

is contained in an

interval $I_1 = [a_1, b_1]$

We set $n_1 = 1$.

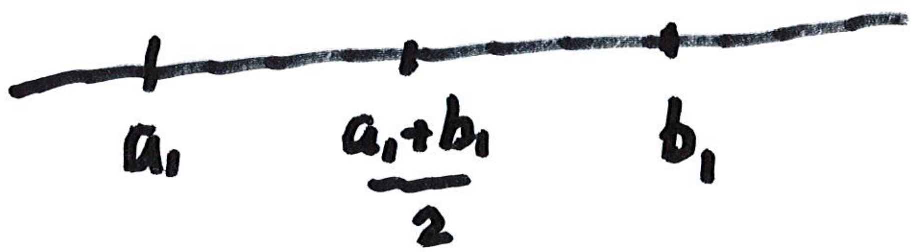
We now bisect I_1 into

two intervals I_1' and I_1''.

More precisely,

$$I_1' = \left[a_1, \frac{a_1+b_1}{2}\right] \qquad \text{and}$$

$$I_1'' = \left[\frac{a_1+b_1}{2}, b_1\right].$$



We divide $\{n \in \mathbb{N} : n > n_1\}$

into two sets,

$$A_1 = \{n \in \mathbb{N} : n > n_1, \ x_n \in I_1'\}$$

$$B_1 = \{n \in \mathbb{N} : n > n_1, \ x_n \in I_1''\}$$

one of which is infinite.

In fact $A_1 \cup B_1$ contains every element of N except for n with $1 \leq n \leq n_1$.

According to our construction,

$$\{ n \in N : n > n_2 ,\ x_n \in I_2 \}$$

is infinite.

If A_1 is infinite, then

we set $I_2 = I_1'$, and

we let n_2 be the smallest

natural number in A_1. Note

that $x_{n_2} \in I_2$.

If A_1 is a finite set, then

B_1 must be infinite,

we let n_2 be the smallest

natural number in B_1, and

we set $I_2 = I_1''$.

We now bisect I_2 into subintervals I_2' and I_2''

and we divide the set

$$\{n \in N : n > n_2,\ x_n \in I_2\}$$

into 2 parts:

$$A_2 = \{n \in N : n > n_2,\ x_n \in I_2'\}$$

$$B_2 = \{n \in N : n > n_2,\ x_n \in I_2''\}$$

If A_2 is infinite, we

take $I_3 = I_2'$, and we let n_3 be the smallest natural number in A_2. If A_2 is a finite set, then B_2 must be infinite, and we take $I_3 = I_2''$, and we let n_3 be the smallest natural number in B_2. Note that $x_{n_3} \in I_3$.

We continue in this way to obtain a sequence of nested intervals

$I_1 \supset I_2 \supset \dots \supset I_k \supset \dots$

and we obtain a subsequence $\{x_{n_k}\}$ of X such that

$x_{n_k} \in I_k$ for $k \in \mathbb{N}$.

In addition, for each k, the set

$$\left\{ n \in \mathbb{N} : n > n_k ,\ x_n \in I_k \right\}$$

is infinite. This fact guarantees that when we split the interval I_k into I_k' and I_k'', one of the corresponding sets is nonempty.

By the Nested Interval Property, there is a point η such that

$$\eta \in \bigcap_{k=1}^{\infty} I_k .$$

The length of I_k is

$\dfrac{(b-a)}{2^{k-1}}$. Since both

x_{n_k} and η both lie in I_k,

it follows that

$$|x_{n_k} - \eta| \le \frac{(b-a)}{2^{k-1}},$$

which implies that the subsequence $\{x_{n_k}\}$ of X converges to η.

Archimedean Property.

1. If $x > 0$, then there exists $n_x \in \mathbb{N}$ so that $x < n_x$.

Pf. Suppose this is NOT true. Then for every $n \in \mathbb{N}$, we would have $n \leq x$, for all n in $\mathbb{N}$. By the Completeness Property, $\mathbb{N}$ has a supremum u.

Then $u-1$ is not an upper bound of $\mathbb{N}$, so there is an integer $m \in \mathbb{N}$ with $u-1 < m$. Adding 1, we get $u < m+1$. This contradicts the fact that $n \leq x$ for all n. Hence, there is an integer n_x with $n_x > x$.

2. For any $\varepsilon > 0$, there is an integer K in $\mathbb{N}$ so that $\frac{1}{n} < \varepsilon$, for all $n \geq K$.

Pf. Set $x = \frac{1}{\varepsilon}$. We showed above that there is an integer n_x, such that $n_x > x$. If we set $K = n_x$, then if $n \geq K$, then

$$n \geq n_x > x = \frac{1}{\varepsilon}.$$

3. If $y > 0$, then there exists $n_y \in N$ such that

$$n_y - 1 \leq y \leq n_y \qquad (*)$$

Pf. The Archimedean Property implies that the subset $E_y = \{ m \in N : y < m \}$ is nonempty. The Well-Ordering Property implies that E_y has a least element, we denote by n_y. Then

$n_y - 1$ does not belong to E_y

Hence we have

$$n_y - 1 \leq y < n_y$$

Density Theorem.

If x and y are any real numbers with $x < y$, then there is a rational number $r \in \mathbb{Q}$ such that $x < r < y$

Pf. We can assume that $x > 0$. (Let $m \in \mathbb{N}$ satisfy $m + x > 0$. Then replace x with $x + m$ and y with $y + m$)

Since $y-x > 0$, it follows from 2. that there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < y-x$.

which gives $nx+1 < ny$.

If we apply (*) to nx, we obtain $m \in \mathbb{N}$ with

$$m-1 \leq nx < m.$$

Therefore,

$$m \leq nx + 1 < ny,$$

which leads to

$$nx < m < ny.$$

Thus the rational number

$r = m/n$ satisfies

$$x < r < y$$

Another Problem using the Monotone Sequence Theorem

Ex. # 2., page 77.

Let $x_1 > 2$ and $x_{n+1} = 2 - \frac{1}{x_n}$

Find $\lim (x_n)$.

First, note that if $x_n > 1$,

then $\frac{1}{x_n} < 1$, so that

$x_{n+1} = 2 - \frac{1}{x_n} > 1$. Hence

$x_n > 1$ for all $n = 1, 2, \dots$.

We want to show that (x_n) is decreasing. We have

$$x_1 - x_2 = x_1 - \left(2 - \frac{1}{x_1}\right) = \frac{(x_1 - x_2)^2}{2} > 0.$$

Similarly, we have:

$$x_{n+1} - x_{n+2} = \left(2 - \frac{1}{x_n}\right) - \left(2 - \frac{1}{x_{n+1}}\right)$$

$$= \left(\frac{1}{x_{n+1}} - \frac{1}{x_n}\right) = \frac{x_n - x_{n+1}}{x_n x_{n+1}}$$

$$< (x_n - x_{n+1})$$

where the final inequality follows from $X_n > 1$, $X_{n+1} > 1$ for all n.

It follows from the Monotone Convergence Theorem that $\tilde{x} = \text{limit}\,(X_n)$ exists, which implies that $\lim(X_n) \geq 1$.

We conclude that $\tilde{x} = 2 - \frac{1}{\tilde{x}}$, which yields that

$$(\tilde{x}-1)^2 = 0, \text{ i.e } \tilde{x} = 1$$