# 3.5 Cauchy's Criterion.

Def'n. A sequence $X = (x_n)$ is a Cauchy sequence if for all $\varepsilon > 0$, there exists a number $H$ in $\mathbb{N}$ so that if $n, m \geq H$, then

$|x_n - x_m| < \varepsilon$

Even though the definition does not mention a limit $x$,

Still, the numbers $x_n$ and $x_m$ get closer as $n, m \to \infty$

---

Lemma. If a sequence approaches a limit $x$, then the sequence $(x_n)$ is Cauchy

Proof of Lemma. If $x = \lim(x_n)$, then given $\varepsilon > 0$, there is a natural number $K$, such that if $n \geq K$, then $|x_n - x| < \frac{\varepsilon}{2}$.

Thus, if $n, m \geq K$, then we have

$$|x_n - x_m| = |(x_n - x) + (x - x_m)|$$

$$\leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

it follows that $(x_n)$ is a

Cauchy sequence.

**Lemma.** A Cauchy sequence is bounded.

Pf. Let $X = (x_n)$ be Cauchy, and set $\varepsilon = 1$. There is $H$ in $\mathbb{N}$ so that, if $n \geq H$, then

$$|x_n - x_H| < 1.$$

By the Triangle Inequality, we have

$$|x_n| \leq |x_H + (x_n - x_H)|$$

$$\leq |x_H| + 1$$

If we set

$$M = \max\left\{ |x_1|, |x_2|, \dots |x_{H-1}|, |x_H| + 1 \right\},$$

then it follows that

$$|x_n| \le M, \quad \text{for all } n.$$

Cauchy Convergence Thm.

A sequence $X = (x_n)$ is convergent if it is a Cauchy sequence. We have to find $x$ !!

We already showed that if $X$ is convergent, then it is Cauchy. To prove the other direction, suppose $X$ is Cauchy. We showed above that $X$ is therefore bounded. By the Bolzano-Weierstrass theorem, there exists a subsequence

$X' = (x_{n_k})$ of $X$ that

converges to a number $x^*$.

We will show that $\lim x_n = x^*$.

Since $X = (x_n)$ is a Cauchy

sequence, given $\varepsilon > 0$, there

is a natural number $H$

such that if $n, m \geq H$.

then $|x_n - x_m| < \frac{\varepsilon}{2}$. (1)

Since the subsequence

$X' = (x_{n_k})$ converges to $x^*$,

there is a natural number

$K \geq H$ that belongs to the

set $\{n_1, n_2, \dots\}$ such that

$$|x_{n_K} - x^*| < \frac{\varepsilon}{2}$$

Since $K \geq H$, it follows

from (1) with $m = K$ that

$$|x_n - x_K| < \frac{\varepsilon}{2} \quad \text{for } n \geq H.$$

Therefore, if $n \geq H$,

we have

$$|x_n - x^*| = |(x_n - x_K) + (x_K - x^*)|$$

$$\leq |x_n - x_K| + |x_K - x^*|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain that $\lim(x_n) = x^*$.

Ex. The polynomial equation

$x^3 - 5x + 1 = 0$ has a root

$r$ with $0 < r < 1$.

We define an iteration

procedure to construct a

sequence $(x_n)$ that

approaches the root $r$.

We define $x_1$ to be any

number with $0 < x_1 < 1$.

and we define

$$x_n^3 - 5x_{n+1} + 1 = 0,$$

or $$x_{n+1} = \frac{1}{5}(x_n^3 + 1).$$

We can estimate $|(x_{n+2} - x_{n+1})|$

by $|x_{n+2} - x_{n+1}|$

$$= \left| \frac{1}{5}(x_{n+1}^3 + 1) - \frac{1}{5}(x_n^3 + 1) \right|$$

$$= \frac{1}{5} \left| x_{n+1}^3 - x_n^3 \right|$$

$$= \frac{1}{5} \left| x_{n+1}^2 + x_{n+1} x_n + x_n^2 \right| |x_{n+1} - x_n|$$

$$\leq \frac{3}{5} |x_{n+1} - x_n| .$$

Definition. We say that a sequence $(x_n)$ of real numbers is contractive if there is a constant $C$, $0 < C < 1$, such that

$$|x_{n+2} - x_{n+1}| \leq C |x_{n+1} - x_n|$$

for all $n \in \mathbb{N}$. $C$ is the constant of the sequence.

We're using the fact that if $0 \le x_1 \le 1$, then $x_n$ also satisfies $0 \le x_n \le 1$ for all $n \in \mathbb{N}$. (by induction)

We now prove.

Thm. Every contractive sequence is a Cauchy Sequence and therefore is convergent.

Observe that in the above example, $(x_n)$ is contractive with $C = 3/5$

Pf. Using the contractive inequality, we get:

$$
\begin{aligned}
|x_{n+2} - x_{n+1}| &\leq C|x_{n+1} - x_n| \\
&\leq C^2 |x_n - x_{n-1}| \\
&\ \ \vdots \\
&\leq C^n |x_2 - x_1|
\end{aligned}
$$

To show that $(x_n)$ is Cauchy, we have

Hence

$|x_m - x_n|$ (2)

$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n|$

$\leq \left( C^{m-2} + \dots + C^{n-1} \right) |x_2 - x_1|$

$= C^{n-1} \left( \frac{1 - C^{m-n}}{1 - C} \right) |x_2 - x_1|$

$\leq C^{n-1} \left( \frac{1}{1 - C} \right) |x_2 - x_1|$

We conclude that $(x_n)$

is a Cauchy sequence

and therefore convergent.

Observe that we can estimate the accuracy of $(x_n)$:

$$|x_m - x_n| \le \frac{c^{n-1}}{1-c} |x_2 - x_1|$$

Since $\lim (x_m) = \tilde{x}$, we have

$$\tilde{x} - x_n \le \frac{c^{n-1}}{1-c} |x_2 - x_1|$$

Similarly,

$$x_m - x_n \geq -\left[\frac{C^{n-1}}{1-C}\,|x_2 - x_1|\right]$$

Letting $x_m \to \tilde{x}$, we obtain

$$\tilde{x} - x_n \geq -\left[\frac{C^{n-1}}{1-C}\,|x_2 - x_1|\right].$$

Combing the inequalities, we get the inequality

$$|\tilde{x}_n - x_n| \leq \frac{C^{n-1}}{1-C}\,|x_2 - x_1|,$$

which shows that the error $\to 0$ exponentially.

Since $x_{n+1} = \frac{1}{5}\left(x_n^3 + 1\right)$

we can take limit as $n \to \infty$:

The limit of $(x_n) = x$, so

$$x = \frac{1}{5}\left(x^3 + 1\right)$$

$$\to \quad 5x = x^3 + 1, \quad \text{or}$$

$$x^3 - 5x + 1 = 0$$