The Sequential Criterion gives an easier way to prove various theorems about limits of functions.

Sequential Criterion. Let $f: A \to \mathbb{R}$ and let c be a cluster point of A. The following statements are equivalent:

(i) $\lim\limits_{x \to c} f = L$.

(ii) For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all n, the sequence converges to L.

Pf. of (i) $\rightarrow$ (ii). Let $\varepsilon > 0$. Then there exists $\delta > 0$ so that if if $0 < |x-c| < \delta$ and $x \in A$, then $|f(x) - L| < \varepsilon$.

Since $\lim\limits_{n\to\infty} (x_n) = c$, there is a $K \in \mathbb{N}$, such that if $n > K$, then $0 < |x_n - c| < \delta$. But for such x_n, we have $|f(x_n) - L|\ \varepsilon$. Hence $(f(x_n))$ converges to L, which proves (ii)

We prove (ii) $\to$ (i) by contradiction. Thus suppose (i) is not true. Then there is an $\varepsilon_0 > 0$ such that no matter how small δ is, there will be at least one $x_\delta \in A$ with $0 < |x_\delta - c| < \delta$ such that $|f(x_\delta) - L| \geq \varepsilon_0$

Hence for every $n \in N$, there is an $x_n \in A$ with $0 < |x_n - c| < \delta$ but such that $|f(x_n) - L| \geq \varepsilon_0$

We now show how the Sequential Criterion can be used to prove the Limit Theorems. We consider the Product Rule. Suppose for example that (x_n) is any sequence in A with $x_n \neq c$ and $c = \lim_{n\to\infty} (x_n)$. It follows that

$$\lim_{n\to\infty} f(x_n) = L \text{ and } \lim_{n\to\infty} g(x_n) = M.$$

By definition,

$$(fg)(x_n) = f(x_n)\, g(x_n)$$

By the Product Rule for sequences

$$\lim \left(f(x_n)\, g(x_n) \right) = \left[\lim \left(f(x_n) \right) \right] \cdot \left[\lim \left(g(x_n) \right) \right]$$

$= LM$. Hence, we have proved

that

$$\lim_{n\to\infty} (fg) = \lim_{n\to\infty} (f) \lim_{n\to\infty} (g) = LM.$$

We give another example.

Thm. Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A. If

$$a \leq f(x) \leq b \quad \text{for all } x \in A,\ x \neq c.$$

and

If $\lim\limits_{n\to\infty} f$ exists, then

$$a \leq \lim_{n\to\infty} f \leq b.$$

Pf. If $\lim\limits_{n\to\infty} f$, then the

Seq. Crit. implies that if (x_n) is any sequence in A such that $x_n \neq c$, and if (x_n) converges to c, then $(f(x_n))$ converges to L. Since $a \leq f(x_n) \leq b$ for all $n \in \mathbb{N}$, the sequential version implies that $a \leq L \leq b$.

We now prove a functional analog of the Squeeze Thm.

Squeeze Theorem for functions

Let $A \subseteq \mathbb{R}$, let $f, g, h : A \to \mathbb{R}$, and let c be a cluster point of A. If

$$f(x) \leq g(x) \leq h(x) \quad \text{for all } x \in A, \text{ with } x \neq c,$$

and if $\lim_{x \to c} f(x) = L = \lim_{x \to \infty} h(x)$,

then $\lim_{n \to \infty} g = L$.

Pf. Let (x_n) be any sequence in A

with $x_n \neq c$ and $\lim\limits_{n\to\infty} x_n = c$.

Then the Seq. Crit. implies

that $\lim\limits_{n\to\infty} f(x_n) = L = \lim\limits_{n\to\infty} h(x_n)$.

Since $f(x_n) \leq g(x_n) \leq h(x_n)$,

it follows from the Seq. Crit.

that $\lim\limits_{n\to\infty} g(x_n) = L$. But

this is true for any sequence (x_n)

converging to c with $x_n \neq c$,

so we conclude that

$$\lim_{x \to c} g = L,$$ which proves

the Sequence Thm.

Ex. Evaluate $\lim_{x \to 0} \sqrt{x} \sin(1/x^2)$, for $x > 0$.

Note that

$$-\sqrt{x} \leq \sqrt{x} \sin(1/x^2) \leq \sqrt{x}.$$

Since $\lim_{x \to 0} -\sqrt{x} = 0 = \lim_{x \to 0} \sqrt{x}$,

it follows that $\lim_{x \to 0} \sqrt{x} \sin(1/x^2) = 0$.

There are also

Divergence Criteria:

Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$, and let c be a cluster point of A.

(a) If $L \in \mathbb{R}$, then f does NOT have limit L at c if there exists a sequence $(x_n) \in A$ with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence does not converge to L.

(b) We obtain another

Divergence Criterion

by replacing "does not

converge to L" with

does not converge in $\mathbb{R}$

Ex. Let $f(x) = \frac{1}{x}$, for $x \neq 0$.

Let $c = 0$, and set $x_n = \frac{1}{n}$.

Then $f(x_n) = 1 / \frac{1}{n} = n$

does not converge to any L

since (n) is unbounded.

Ex. $g(x) = \sin(1/x)$, and let $c = 0$.

Set $x_n = \dfrac{1}{n\pi + \frac{\pi}{2}}$.

Then $\lim_{x \to 0} \sin(1/x)$ does not exist, since $f(x_n) = 1$ when n is even and -1 when n is odd. If f had a limit at $x = 0$, then f applied to any subsequence would have that value, hence $f(x_n)$ cannot have two limits.

4.3 Some Extensions of the Limit Concept.

Def'n. If $c \in \mathbb{R}$ is a cluster point of $A \cap (c, +\infty) = \{x \in A ; x > c\}$, then we say L is a right-hand limit of f at c, and we write

$$\lim_{x \to c^+} f = L$$

if for any $\varepsilon > 0$, there is $\delta > 0$

such that for all $x \in A$ with

$0 < x - c < \delta$, then

$|f(x) - L| < \varepsilon$.

There is a similar definition

for left-hand limits.

These limits are called

one-sided limits.

There is a Sequential Criterion right-handed (and left-handed) limits.

Thm. Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of $A \cap (c, \infty)$. Then the following statements are equivalent.