Infinite Limits

Def'n. Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$

and let c be a cluster point of A.

We say f tends to ∞ as $x \to c$

and write $\lim_{x \to c} f = \infty$

if for all $\alpha \in \mathbb{R}$, there is $\delta > 0$

so that if $x \in A$ and $0 < |x - c| < \delta$

then $f(x) > \alpha$

Ex. Show $\lim_{x \to 0} \frac{1}{\sqrt{x}} = \infty$. $x > 0$.

We need $\frac{1}{\sqrt{x}} > \alpha$

$$\rightarrow \frac{1}{x} > \alpha^2 \rightarrow x < \frac{1}{\alpha^2}$$

$\therefore$ Set $\delta = \frac{1}{\alpha^2}$.

If $0 < x < \frac{1}{\alpha^2} \rightarrow \sqrt{x} < \frac{1}{\alpha}$

$\rightarrow \frac{1}{\sqrt{x}} > \alpha$ ✓

Limits at ∞.

Def'n. Let $A \subseteq \mathbb{R}$, and let $f: A \to \mathbb{R}$.

Suppose that $(a, \infty) \subseteq A$ for

some $a \in \mathbb{R}$. We say L is a

limit of f as $\to \infty$, and we write

$\lim_{x \to \infty} f(x) = L$ if given any $\varepsilon > 0$

there is $K > a$ so that

if $x > K$, then $|f(x) - L| < \varepsilon$

Ex. Show that $\lim_{x \to \infty} \frac{x^2 - 3x - 1}{2x^2 + 1} = \frac{1}{2}$

It's easy to show that $\lim_{x \to \infty} \frac{1}{x} = 0$.

Note $\frac{x^2 - 3x - 1}{2x^2 + 1} = \frac{x^2\left(1 - \frac{3}{x} - \frac{1}{x^2}\right)}{x^2\left(2 + \frac{1}{x^2}\right)}$

$= \frac{1 - \frac{3}{x} - \frac{1}{x^2}}{2 + \frac{1}{x^2}}$. Using the analogs of the limit rules

$\lim \frac{1}{x} = 0$, $\frac{1}{x^2} = 0$, etc we obtain $\displaystyle\lim_{x \to 0} \frac{1 - \frac{3}{x} - \frac{1}{x^2}}{2 + \frac{1}{x^2}} = \frac{1 - 0 - 0}{2 + 0}$

$= \frac{1}{2}$

Def'n. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in A$. We say f is continuous at c if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if x satisfies $x \in A$ with $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

If f is continuous at c, then three conditions must hold:

(i) f must be defined at c,

(ii) The limit of f at c must exist,

(iii) These two values must be equal.

Of course we have the following result.

Sequential Criterion for Continuity.

A function $f: A \to \mathbb{R}$ is continuous at c if and only if for every sequence (x_n) in A that converges to c, the sequence $f(x_n)$ converges to $f(c)$

converges to c, the sequence $(f(x_n))$ converges to $f(c)$.

And we have a

Discontinuity Thm. Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$, and let $c \in A$. Then f is discontinuous at c if and only if there exists a sequence (x_n) in A such that (x_n) converges to c, but the sequence $(f(x_n))$ does NOT converge to $f(c)$.

Pf. If f is discontinuous at c.

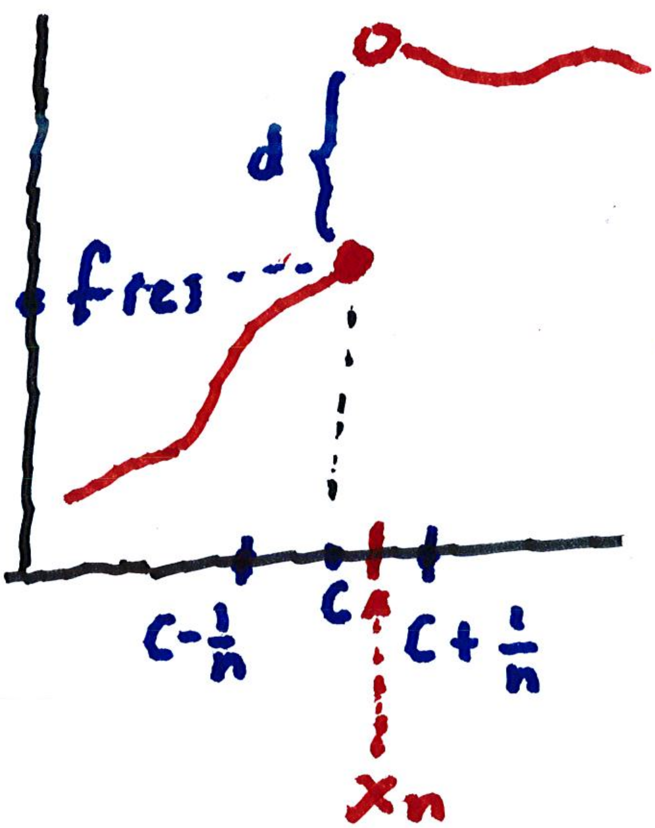
there is a number $\varepsilon_0 > 0$

such that for every $n \in \mathbb{N}$,

there is a number $x_n \in A$

with $|x_n - c| < \frac{1}{n}$ and

$|f(x_n) - f(c)| \geq \varepsilon_0$



Choose $\varepsilon_0 = \frac{d}{2}$

If $\frac{1}{n}$ is sufficiently small,

$|f(x_n) - f(c)| \geq \varepsilon_0$

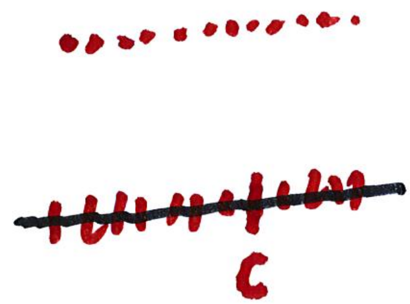
Def'n. Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$.

If B is a subset of A, we say that f is continuous on the set B if f is continuous at every point of B.

Ex. Let $A = \mathbb{R}$, and define the Dirichlet function f by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

We show that f is discontinuous at every point of $\mathbb{R}$. First, we suppose that c is rational, so that $f(c) = 1$. Let (x_n) be a sequence of irrational numbers that converge to c



Set $x_n = c + \dfrac{\sqrt{2}}{n}$

Then $f(x_n) = 0$ for all $n \in \mathbb{N}$

Since $f(c)=1$, it follows that $f(x_n)$ does not converge to $f(c)$. By the Discontinuity Criterion f is not continuous at c.

Similarly, suppose c is an irrational number. Since the rationals are dense in $\mathbb{R}$, for every n we can find a rational number $x_n \in (c, c+\frac{1}{n})$, so that $\lim_{n\to\infty} f(x_n) = 1$.

Since (x_n) converges to c,

and $f(c) = 0$, it follows that

$\lim_{n\to\infty} (f(x_n))$ does not converge

to $f(c)$. By the Discontinuity

Criterion, it follows that

f is discontinuous at c.

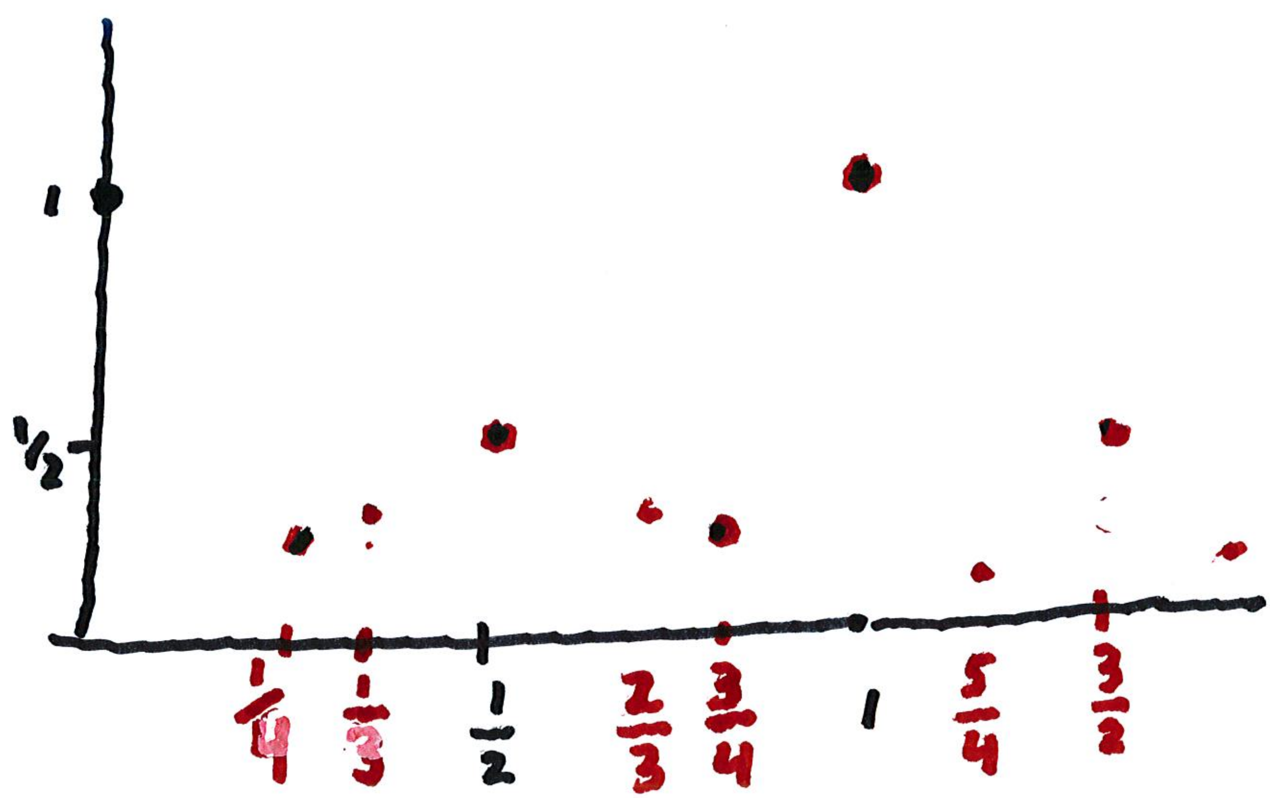
$\therefore$ f is discontinuous at each

point of $\mathbb{R}$.

Ex. Thomae Fcn. We define

$f : [0, \infty) \to \mathbb{R}$ by

$$h(x) = \begin{cases} 1/q & \text{if } x = p/q \text{ and } p, q \text{ have} \\ & \text{no common factor} > 1 \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$



We show that the function h is continuous at each irrational number x' and discontinuous at each rational number x''.

It's easy to show that h is discontinuous at each rational.

In fact, as above, if c is rational, then $h(c) = 1/q$ for some positive q. But let

(x_n) be a sequence of irrational numbers that converges to c. Then $f(x_n)$ does NOT converge to $f(c)$.

Hence the Discontinuity Thm implies that h is discontinuous at c.

Now we show h is continuous at each irrational number b.

Let ε be any positive. Then there is a number n_0 with $\frac{1}{n_0} < \varepsilon$. There are only a finite number of rationals with denominator less than n_0 in the interval $(b-1, b+1)$.

Hence we can choose $\delta > 0$ so small that the neighborhood $(b-\delta, b+\delta)$ contains no rational numbers with denominator less than n_0. It follows that for $|x-b| < \delta$, $x \in A$ we have

$$|h(x) - h(b)| = |h(x)| \leq \frac{1}{n_0} < \varepsilon.$$

Thus h is continuous at the irrational number b

Ex. Consider the function

$g(x) = \sin(1/x)$. We showed

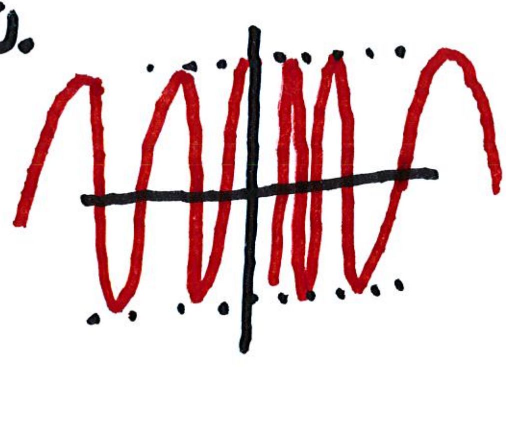
$\lim_{x \to 0} \sin(1/x)$ does not exist.

so g could never be made

to be continuous at 0.

But $G(x) = \begin{cases} x\sin(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

is continuous at 0.



(not a 'jump' discontinuity)