5.3 Continuous Functions on Intervals

Def'n. A function $f: A \to \mathbb{R}$ is said to be bounded on A if there is a constant $M > 0$ such

$|f(x)| \leq M$, for all $x \in A$.

Ex $f(x) = \frac{1}{x}$ is not bounded on $(0, 1]$

Thm. Let $I = [a,b]$ and let $f: I \to \mathbb{R}$ be continuous on I. Then f is unbounded on I.

Pf. (by Contradiction)

Suppose f is not unbounded.

Then for every integer $n \in \mathbb{N}$, there is a point x_n in I such that $|f(x_n)| > n$.

Since I is bounded, the

the sequence $X = (x_n)$ is bounded. Hence the Bolzano-Weierstrass Theorem implies there is a subsequence $X' = (x_{n_r})$ of X that converges to a number x. Since I is closed and the elements of X' belong to I, it follows that $x \in I$.

Then f is continuous at x, so that $(f(x_{n_r}))$ converges to $f(x)$. We then conclude that the convergent sequence $(f(x_{n_r}))$ must be bounded. But this is a contradiction since

$|f(x_{n_r})| > n_r \geq r$, all $r \in \mathbb{N}$

Def'n. Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. We say that has an absolute maximum on A if there is a point $x' \in A$ so that

$$f(x') \geq f(x), \qquad \text{for all } x \in A.$$

Similarly, f has an absolute minimum on A if there is a point $x'' \in A$ such that

$$f(x'') \leq f(x), \qquad \text{for all } x \in A.$$

Maximum Minimum Theorem.

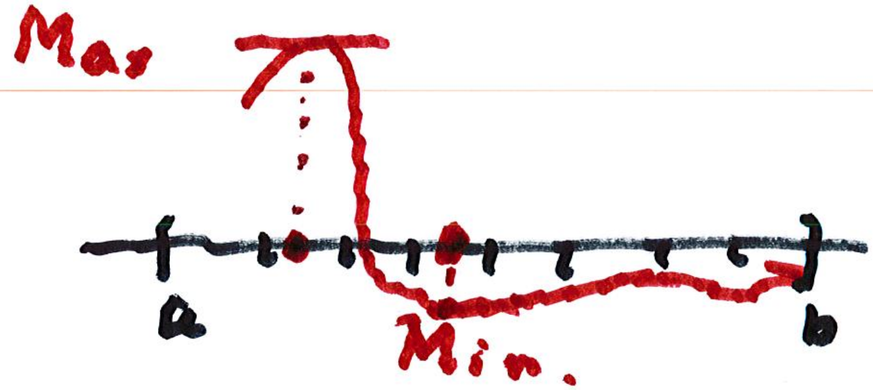
Let $I = [a, b]$ and let

$f: I \to \mathbb{R}$ be continuous on I.

Then f has an absolute

maximum and an absolute

minimum on I



Proof: Consider the set $f(I) = \left\{ f(x) : x \in A \right\}$.

Let $s' = \sup f(I)$ and let $s'' = \inf$ [illegible] We will show that there exist points x' and x'' such that $s' = f(x')$ and $s'' = f(x'')$

We do this for x'.

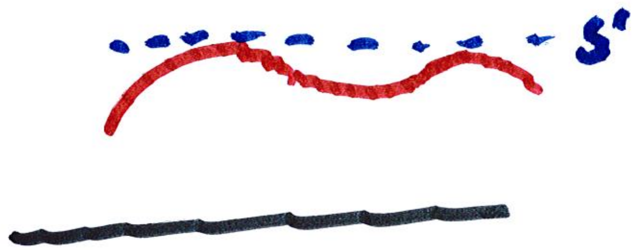
Since $S' = \sup f(I)$,

We want to show that

there exists at least one

number x' so that

$f(x') = S'$.



If not,

there exists no number x

satisfying $f(x) = S'$.

Consider the function

$$g(x) = \frac{1}{s' - f(x)}.$$

We already know $f(x) \leq s'$

for all x and and that there

is no number x with $f(x) = s'$

Hence $s' > f(x)$, for all $x \in I$.

Hence the function

$$g(x) = \frac{1}{s' - f(x)} > 0 \quad \text{for all } x \in I.$$

The function g is continuous at each point $x \in I$. By the above result there is a number M so that

$$\frac{1}{s' - f(x)} < M$$

$$\to \quad M < s' - f(x)$$

$$\to \quad f(x) < s' - M.$$

which shows that $s' - M$ is an upper bound, which shows

Our third theorem is:

Location of Roots Thm.

If $I = [a, b]$, let $f: I \to \mathbb{R}$ be continuous on I. If $f(a) < 0 < f(b)$ (or $f(a) > 0 > f(b)$) then there exists a number $c \in (a, b)$ so that $f(c) = 0$.

Proof. We assume that $f(a) < 0 < f(b)$. Let $I_1 = [a_1, b_1]$, where $a_1 = a$, and $b_1 = b$. We let

$$p_1 = \frac{a_1 + b_1}{2}.$$

If $f(p_1) = 0$, we take $c = p_1$ and we are done. If $f(p_1) \neq 0$, then either, $f(p_1) > 0$ or $f(p_1) < 0$.

In the first case, set $a_2 = a_1$
$(f(p_1) > 0)$

and set $b_2 = p_1$.

Then $f(a_2) < 0$ and $f(b_2) > 0$

We continue the bisection process. Assume intervals $I_1 \supset I_2, \ldots, I_k$ have been obtained by successive bisection. We have $f(a_k) < 0 < f(b_k)$.

We set $p_k = \frac{1}{2}(a_k + b_k)$.

If $p_k = 0$, we take $c = p_k$ and we are done. If $f(p_k) > 0$, we set $a_{k+1} = a_k$ and $b_{k+1} = p_k$.

If $f(p_k) < 0$, we set $a_{k+1} = p_k$ and $b_{k+1} = b_k$.

In either case, we let $I_{k+1} = [a_{k+1}, b_{k+1}]$

Then $I_{k+1} \subset I_k$ and

$f(a_{k+1}) < 0$ and $f(b_{k+1}) > 0$.

If the process terminates by locating a point p_n such that $f(p_n) = 0$, then we are done. If the process does not terminate, then we have a nested sequence of closed bounded intervals $I_n = [a_n, b_n]$ such that such that

$f(a_n) < 0 < f(b_n)$.

The intervals are obtained by repeated bisection, so that the length of I_n equals $b_n - a_n = \frac{b-a}{2^{n-1}}$.

Let c be the point belongs to I_n for all n.

Since $a_n \le c \le b_n$,

we have

$$0 \le c - a_n \le b_n - a_n = \frac{(b-a)}{2^{n-1}},$$

and

$$0 \le b_n - c \le b_n - a_n = \frac{(b-a)}{2^{n-1}}.$$

The Squeeze Theorem implies

that $\lim(a_n) = c = \lim(b_n)$

Since f is continuous at c,

we have

$\lim(f(a_n)) = f(c) = \lim(f(b_n))$.

The fact that $f(a_n) < 0$

for all $n \in \mathbb{N}$ implies that

$f(c) = \lim(f(a_n)) \leq 0$.

Also, the fact $f(b_n) > 0$

implies that

$f(c) = \lim(f(b_n)) \geq 0$.

We conclude that $f(c) = 0$.

Bolzano's Intermediate Value Thm:

Suppose that I is an interval and let $f: I \longrightarrow \mathbb{R}$ be continuous on I. If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then there exists a point c with $a < c < b$ such that $f(c) = k$.

Pf. Suppose $a < b$,

and let $g(x) = f(x) - k$.

Then the above theorem

implies there is a point c

with $0 = g(c) = f(c) - k$,

which gives $f(c) = k$