Fundamental Theorem of Calculus, Part 1.

Let f be a continuous function on a closed bounded interval J.

Given a number $a \in J$, we define a function F on J as follows: $F(x) = \int_a^x f$, all $x \in J$.

Then F is continuous on J, and at each $x_0 \in J$, F is differentiable and $F'(x_0) = f(x_0)$.

Proof. Since f is continuous on J, it follows that f is bounded, i.e. $|f(x)| \leq M$, if $x \in J$.

Hence, if x and y are two points with, say $x \leq y$. Then

$$F(y) - F(x) = \int_a^y f - \int_a^x f = \int_x^y f,$$

so that

$$|F(y) - F(x)| = \left| \int_x^y f \right| \leq \int_x^y |f|$$

$$\leq \int_x^y M = M(y-x)$$

Thus, f is Lipschitz on $[c, d]$

which implies that F is uniformly continuous on J.

Now suppose that f is right-continuous at x_0, where $x_0 \in J$. Consider $x \in J$ with

$x > x_0$. Then

$$F(x) - F(x_0) = \int_{x_0}^{x} f(t)\,dt$$

and
$$f(x_0) = \frac{1}{x - x_0} \int_{x_0}^{x} f(x_0)\,dt.$$

From these two equations we get

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)$$
$$= \frac{1}{x - x_0} \int_{x_0}^{x} [f(t) - f(x_0)]\,dt$$

and thus,

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right|$$

$$\leq \frac{1}{x - x_0} \int_{x_0}^{x} |f(t) - f(x_0)| \, dt \, .$$

Let $\varepsilon > 0$ be given. Since f is

right-continuous at x_0, there

exists a $\delta > 0$ so that for all $t \in J$,

$x_0 < t < x_0 + \delta \Rightarrow |f(t) - f(x_0)| \, dt.$

Thus, if $x_0 < x < x_0 + \delta$, then

$$\int_{x_0}^{x} |f(t) - f(x_0)| \, dt$$

$$\leq \int_{x_0}^{x} \varepsilon \, dt = \varepsilon (x - x_0),$$

so that

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| \leq \varepsilon.$$

This proves that

$$\lim_{x \to x_0^+} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0).$$

Similarly, if f is left-continuous at x_0, then it can be shown that

$$F'(x_0^-) = f(x_0).$$

It follows that if f is continuous at x_0 in the usual two-sided sense, then F is differentiable at x_0 in the usual two-sided sense and

$$F'(x_0) = f(x_0).$$

Corollary. If f is continuous on J, then f has an antiderivative F on J.

To say that F is an antiderivative, this means $\underline{F'(x) = f(x)}$, for all $x \in J$

Fundamental Theorem of Calculus. Part 2.

Suppose f is differentiable at each $x \in J$. Assume also that f' is continuous on a closed bounded interval.

~~Then~~ If $[a,b] \subset J$, then

$$f(b) - f(a) = \int_a^b f'(t)\,dt.$$

Pf. We show that for all $x \in [a,b]$,

$$f(x) - f(a) = \int_a^x f'(t)\,dt . \qquad (1)$$

Note that the left-handed side of (1) is $f'(x)$, { where $x \in J$.}

Part 1 of the Fundamental Theorem of Calculus implies that the derivative of the right-hand side exists and equals $f'(x)$. Note that one of the corollaries of the Mean Value Thm implies that since both functions have the same derivative, it must be that these two functions differ by a ~~function~~ constant, i.e.,

$$f(x) - f(a) = \int_a^x f'(t)\,dt + C$$

If we set $x = a$, we obtain $0 = 0 + C$. It follows that

$$f(x) - f(a) = \int_a^x f'(t)\,dt.$$

Setting $x = b$, we obtain

$$F(b) - F(a) = \int_a^b f'(t)\,dt$$