Complex Numbers

A complex number is a number of the form $z = x + yi$, where x and y are real numbers and i satisfies $i^2 = -1$. It is obvious how to add complex numbers:

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$.

For multiplication we have

$$z_1 z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2)$$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

All of the standard properties of a field are satisfied:

Addition and Multiplication are commutative and associative.

A number $z = x + iy$ also has a multiplicative inverse, namely: $(x+yi)^{-1} = \dfrac{x-yi}{x^2+y^2}$.

Also the distributive property holds:

$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.

However there are no order relations. One cannot say $z_1 < z_2$.

We define limits of complex functions as follows: If z_n, $n=1,2,\ldots$, is a sequence of complex numbers, then we say

$\lim_{n\to\infty} z_n = w$ if for every $\varepsilon > 0$, there is an integer $N(\varepsilon)$, such that if $n \geq N(\varepsilon)$,

$$|z_n - w| < \varepsilon.$$

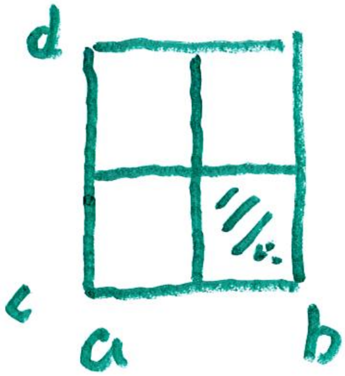
There is a version of the Bolzano - Weierstrass Thm.

Theorem. Suppose there is a sequence $(z_n ; n = 1, 2 \ldots)$ in the rectangle

$$R = \left\{ (x + yi) ; \quad a \leq x \leq b, \text{ and } c \leq y \leq d \right\}$$

Then there is a subsequence

$\{z_{n_r};\ r = 1, 2, \dots\}$ that converges to a number $z' \in R$.



To prove this, there must be an infinite number of the complex numbers in one of the 4 rectangles obtained by bisecting $[a,b]$ and $[c,d]$.

Continue this argument, so that each successive rectangle has infinitely many elements of the original sequence. By the Nested Interval Property, one obtains a sequence $z_{n_r} = x_{n_r} + i y_{n_r}$ that converges to a complex number $z' \in R$.

Suppose that $f: R \to \mathbb{C}$

that is continuous, i.e.,

$$\lim_{z \to z_0} f(z) = f(z_0), \quad \text{for all } z \in R.$$

Then the Bolzano-Weierstrass

Theorem implies that there

is a number m such that

$$|f(z)| \leq m, \qquad \text{for all } z \in R$$

Furthermore, there are complex numbers z_0 and z_1 in R such that

(1) $|f(z_0)| \leq |f(z)|$, for all $z \in R$

and

(2) $|f(z_1)| \geq |f(z)|$, for all $z \in R$.

We will use (1) in our proof of the Fundamental Theorem of Algebra.

One more algebraic property.

Given $z \in \mathbb{C}$, we can write

$$z = r(\cos\theta + i\sin\theta).$$

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

then by the multiplication

formula, one obtains

$$z_1 z_2 = r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)$$

By induction, one easily obtains de Moivre's Formula:

If $z = r(\cos\theta + i\sin\theta)$, then

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

We can easily compute n-th roots. If $w = R(\cos\phi + i\sin\phi)$, then $z = R^{1/n}\left(\cos\left(\frac{\phi}{n}\right) + i\sin\left(\frac{\phi}{n}\right)\right)$ satisfies $z^n = w$.

Now we prove:

Fundamental Theorem of Algebra. Give any polynomial

$f(z) = a_n z^n + \dots + a_1 z + a_0$, $a_n \neq 0$,

with complex coefficients and $n \geq 1$, there is a z_0 such that

$f(z_0) = 0$

Proof. The function $f(z)$

$= z^n + a_{n-1} z^{n-1} + \dots + a_0$

is continuous. If we write

$z = x + iy$ and take k-th powers,

such as $(x+iy)^k$, one can

verify that the real part

is a polynomial as is the

imaginary part. Also, by

using the composition of

continuous functions is also

continuous.

If $E(z) = a_{n-1} z^{n-1} + \cdots + a_0$,

we want to estimate $|E(z)|$.

Let $A = \mathrm{Max}(|a_0|, \ldots, |a_{n-1}|)$.

If $|z| \geq 1$, then

$$|E(z)| \leq |a_{n-1} z^{n-1} + \cdots + a_0|$$

$$\leq nA|z|^{n-1} \leq \frac{|z|^n}{2},$$

if $|z| \geq 2nA$.

Summing up, if $|z| \geq \mathrm{Max}\{1, 2nA\}$

then $|E(z)| \leq \dfrac{|z|^n}{2}$.

We have shown that if

$|z| \leq M = \max(1, 2nA)$, then

$$|f(z)| = |z^n + E_n(z)|$$

$$\geq |z^n| - \frac{|z|^n}{2} = \frac{|z^n|}{2}.$$

In particular, if $|z| \geq M$

and also $|z| \geq \sqrt[n]{2|f(0)|}$, then

$$|f(z)| \geq \frac{|z|^n}{2} = \frac{2|f(0)|}{2} = |f(0)|$$

Now let $[a, b] \times [c, d]$

be a closed rectangle which

contains $\left\{ z : |z| \leq \max\left(M, \sqrt[n]{2f(0)}\right) \right\}$,

and suppose that the minimum

of $|f|$ on $[a,b] \times [c,d]$ is

attained at z_0, so that

(1) $|f(z_0)| \leq |f(z)|$ for z in

$[a,b] \times [c,d]$.

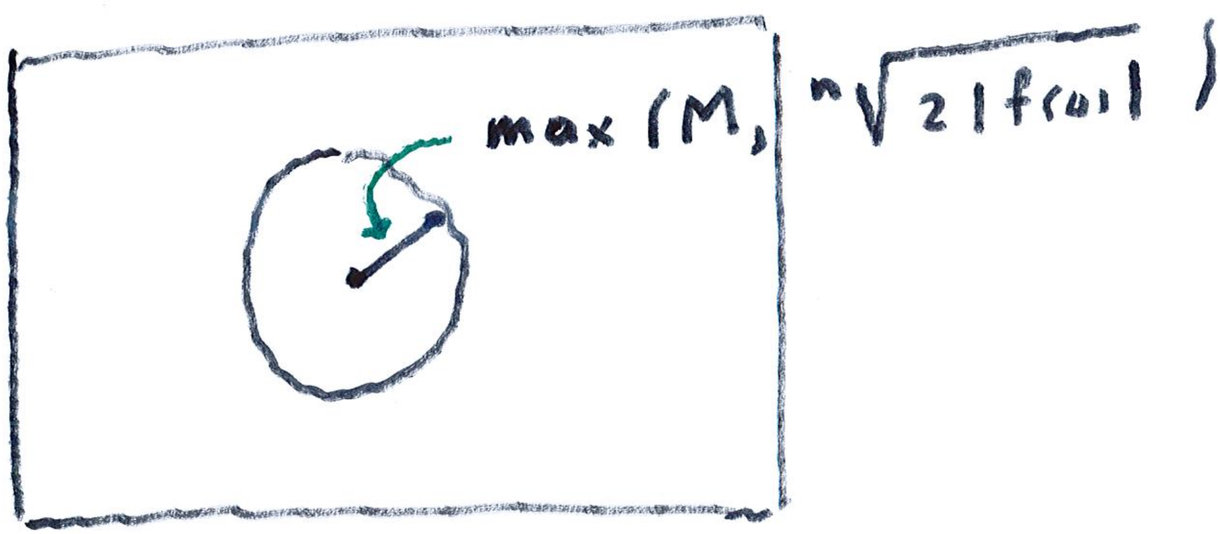
It follows, in particular,

that $|f(z_0)| \le |f(0)|$. Thus

(2) if $|z| \ge \max\left(M, \sqrt[n]{2|f(0)|}\right)$,

then

$$|f(z)| \ge |f(z)| \ge |f(0)| \ge |f(z_0)|.$$



Combining (1) and (2), we see that $|f(z)| \geq |f(z_0)|$, for all z.

To complete the proof, we show that $f(z_0) = 0$. It is convenient to [illegible] the function g by

$$g(z) = f(z + z_0).$$

Then g is a polynomial of

degree n whose minimum absolute value occurs at 0.

Suppose instead that $g(0) = \alpha \neq 0$. If m is the smallest positive power of z which occurs in the expression for g, we write

$$g(z) = \alpha + \beta z^m + c_{m+1} z^{m+1} + \dots + c_n z^n.$$

where $\beta \neq 0$.

As noted above there is a complex number γ such that

$$\gamma^m = -\frac{\alpha}{\beta}.$$

Then setting $d_k = c_k \gamma^k$, we have

$$\left| \alpha + \beta \gamma^m z^m + d_{m+1} z^{m+1} + \dots \, d_n z^n \right|$$

$$= \left| \alpha - \alpha z^m + d_{m+1} z^{m+1} + \dots \right|$$

$$= \left| \alpha (1 - z^{m} + \frac{d_{m+1}}{\alpha} z^{m+1} + \dots \right|$$

$$= \left| \alpha \left(1 - z^{m} + z^m \left[\frac{d_{m+1}}{\alpha} z + \cdots \right] \right) \right|$$

$$= |\alpha| \left| 1 - z^m + z^m \left[\frac{d_{m+1}}{\alpha} z + \cdots \right] \right|$$

If we choose $|z|$ to be sufficiently small, real and positive, then

$$\left| z^m \left[\frac{d_{m+1}}{\alpha} z + \cdots \right] \right| < |z^m| = z^m.$$

Consequently, if $0 < z < 1$, then

$$\left| 1 - z^m + z^m \left[\frac{d_{m+1}}{\alpha} + \cdots \right] \right|$$

$$\leq |1 - z^m| + \left| z^m \left[\frac{d_{m+1}}{\alpha} + \cdots \right] \right|$$

$$= 1 - z^m + \left| z^m \left[\frac{d_{m+1}}{\alpha} z + \cdots \right] \right|$$

$$< 1 - z^m + z^m = 1$$

This shows that for such z

$|g(\gamma z)| < |\alpha|$, which

is a contradiction. Hence

$f(z_0) = 0$.