This week we prove three theorems: We start with a famous result of Weierstrass.

Thm. Suppose that f is a continuous complex function on $[a, b]$. Then for any $\varepsilon > 0$, there is a polynomial $P(x)$ such that $|P(x) - f(x)| < \varepsilon$ for all $x \in [a, b]$

We can assume that $[a, b] = [0, 1]$.

In fact, if $f(x)$ is any continuous complex function on $[a, b]$, then $F(t) = f(a + (b-a)t)$ is also continuous and complex for $0 \le t \le 1$. If theorem is known for $[0, 1]$, then there is a polynomial $p(t)$ so $|F(t) - p(t)| < \varepsilon$, $t \in [0, 1]$.

Using the substitution $t = \dfrac{x-a}{b-a}$, we get $F\left(\frac{x-a}{b-a}\right) = f(x)$

satisfies

$$\left|f(x) - p\left(\frac{x-a}{b-a}\right)\right| < \varepsilon, \quad x \in [a, b].$$

We can also assume that

$f(0) = f(1) = 0$. For if the

theorem is proved for this case,

consider

$$g(x) = f(x) - f(0) - x\left[f(1) - f(0)\right]$$

$$(0 \le x \le 1).$$

Here $g(0) = g(1) = 0$, so if the

result is true, we obtain

that there is a polynomial

P so

(1) $$|g(x) - P(x)| < \varepsilon, \quad x \in [0,1].$$

Note that $P_1(x) = f(x) - g(x)$ is a

polynomial. Hence (1) implies

$$\left| g(x) + (f(x) - g(x)) - (P(x) + P_1(x)) \right| < \varepsilon.$$

or $$\left| f(x) - (P(x) + P_1(x)) \right| < \varepsilon$$

$$x \in [0,1]$$

This shows f can be approximated

by polynomials to within ε.

Hence if the result is true

when $g(0) = g(1) = 0$, it also

holds in the general case.

Hence we can assume $g(0) = g(1) = 0$.

We define $f(x) = 0$ for $x \notin [0,1]$.

Hence f is uniformly continuous

on the whole line.

We set

$$(2) \quad Q_n(x) = c_n(1-x^2)^n \quad \text{for } n = 1, 2, 3, \ldots$$

where c_n is chosen so that

$$(3) \quad \int_{-1}^{1} Q_n(x)\, dx = 1, \quad n = 1, 2, 3, \ldots$$

We need to estimate the size of c_n. Since

$$\int_{-1}^{1} (1-x^2)^n \, dx = 2\int_0^1 (1-x^2)^n \, dx$$

$$\geq 2\int_0^{1/\sqrt{n}} (1-x^2)^n \, dx$$

$$\geq 2\int_0^{1/\sqrt{n}} (1-nx^2) \, dx = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}.$$

Since $c_n = \left(\int_{-1}^{1} (1-x^2)^n \, dx \right)^{-1} \quad \frac{1}{\sqrt{n}},$

it follows that

(4) $\qquad c_n < \sqrt{n}$

The inequality

$$(1-x^2)^n \geq 1-nx^2$$

which we used above can be proved by considering the function

$$(1-x^2)^n - 1 + nx^2 = h(x)$$

which is zero at $x=0$ and whose derivative is positive in $(0,1)$.

In fact,

$$h'(x) = -2nx(1-x^2)^{n-1} + 2nx$$

$$= 2nx\left\{1-(1-x^2)^{n-1}\right\}.$$

Since $h(x) > 0$ for $x \in (0,1)$,

it follows that

$$(1-x^2)^n \geq 1-nx^2, \quad x \in [0,1).$$

Recall c_n was chosen so

$$c_n \int_{-1}^{1} (1-x^2)^n \, dx = 1$$

For any $\delta > 0$, (2) and (4) imply that

$$(5) \quad Q_n(x) \le \sqrt{n}\,(1-\delta^2)^n, \qquad \text{when } \delta \le |x| \le 1,$$

so that $Q_n \to 0$ uniformly in $\delta \le |x| \le 1$.

Now set

$$P_n(x) = \int_{-1}^{1} f(x+t)\, Q_n(t)\, dt \qquad (\delta \le |x| \le 1).$$

By a simple change of variables, we get

$$P_n(x) = \int_{-x}^{1-x} f(x+t)\, Q_n(t)\, dt$$

$$= \int_{0}^{1} f(t)\, Q_n(t-x)\, dt$$

Note that the last integral is cleary a polynomial in x. Thus $\{P_n\}$ is a sequence of polynomials which are real if f is real.

Given $\varepsilon > 0$, we choose $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \frac{\varepsilon}{2}$.

Let $M = \text{l.u.b.}\ |f(x)|$. Using (3), (5) and the fact that $Q_n(x) \geq 0$, we see that for $0 \leq x \leq 1$,

$$|P_n(x) - f(x)| = \left|\int_{-1}^{1} [f(x+t) - f(x)]\, Q_n(t)\,dt\right|$$

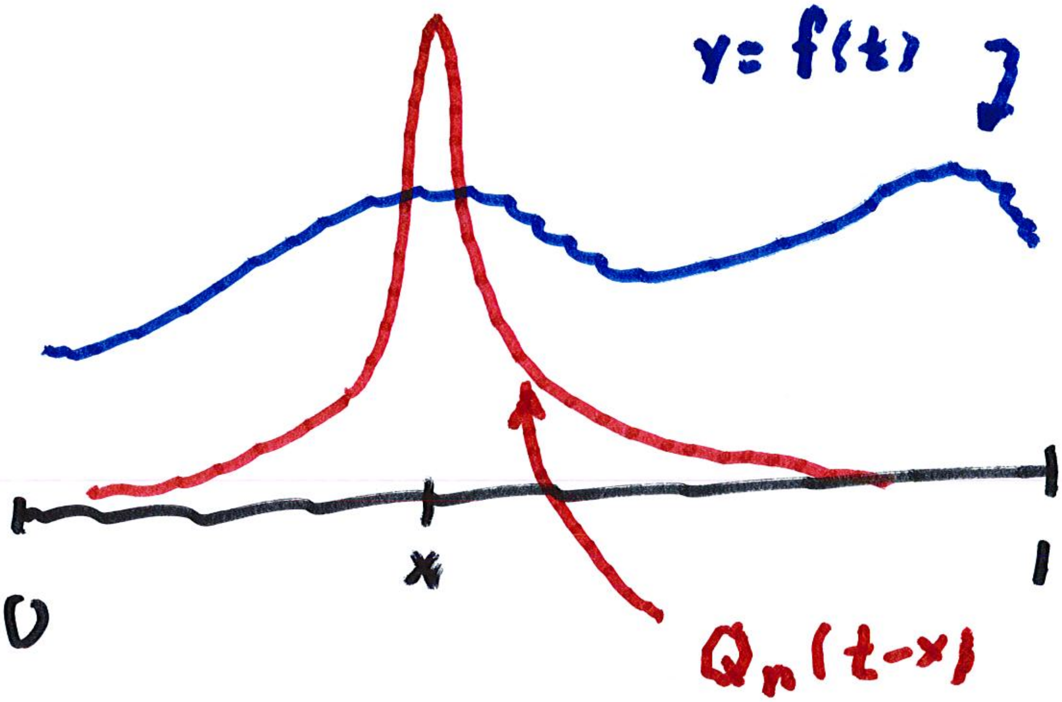
$$\leq \int_{-1}^{1} |f(x+t) - f(x)|\, Q_n(t)\,dt$$

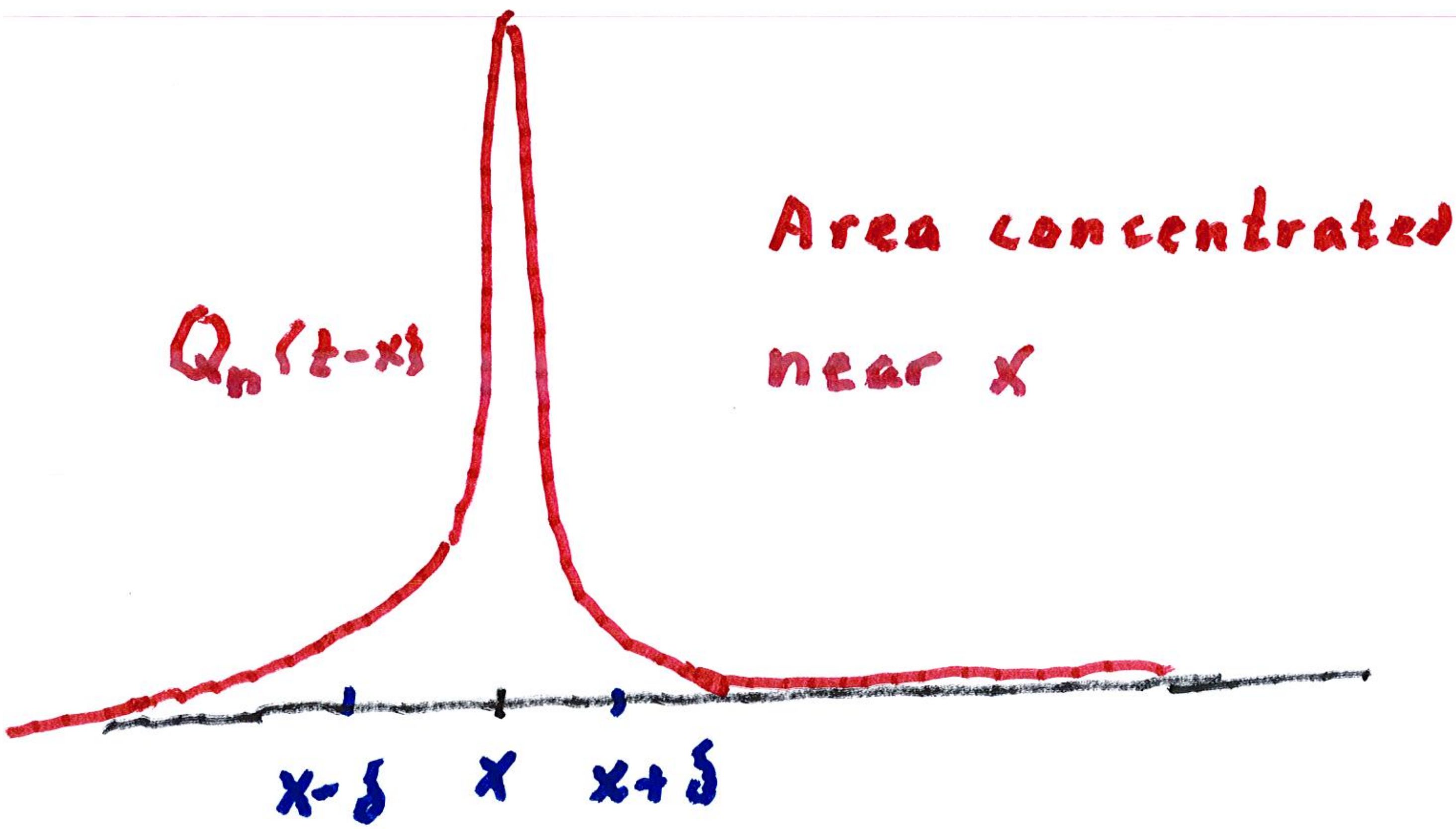
$$\leq 2M \int_{-1}^{-\delta} Q_n(t)\,dt + \frac{\varepsilon}{2} \int_{-\delta}^{\delta} Q_n(t)\,dt$$

$$2M \int_{\delta}^{1} Q_n(t)\,dt$$

$$\leq 4M\sqrt{n}\,(1-\delta^2)^n + \frac{\varepsilon}{2} < \varepsilon$$

for all large enough n, which proves the theorem.


$y = f(t)$
0
x
1
$Q_n(t-x)$


Area concentrated
near x
$Q_n(t-x)$
$x-\delta$
x
$x+\delta$