There is a different formula for the error in Taylor's Formula

Taylor's Thm.

Let $n \in \mathbb{N}$, $I = [a,b]$, and let $f : I \to \mathbb{R}$ such that

$f, f', \ldots, f^{(n)}$ are all continuous on I and that $f^{(n+1)}$ exists on (a,b). If $x_0 \in I$, then for any x in I, there exists

a point c between x and x_0

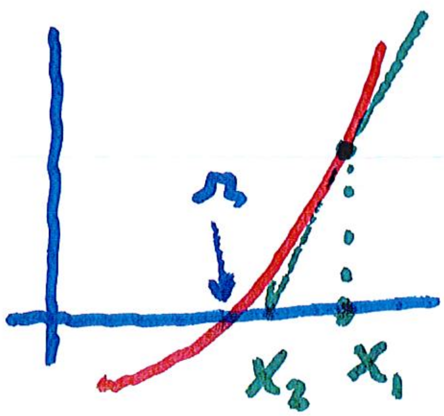
such that

$$f(x) = f(x_0) + f'(x_0)(x-x_0)$$

$$+ \frac{f''(x_0)(x-x_0)^2}{2!} + \dots + \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}$$

$$+ \frac{f^{(n+1)}(c)(x-x_0)^{n+1}}{(n+1)!}$$

6.4.7 Newton's Method



Assume curve is a straight line:

Solve for r:

$$f(x_1) + f'(x_1)(r - x_1) = 0$$

Divide by $f'(x_1)$:

$$\frac{f(x_1)}{f'(x_1)} = x_1 - r$$

$$\rightarrow r = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Set $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\vdots$$

Assuming x_n has been found,

$$x_{n+1} = x_n - \frac{f(x_1)}{f'(x_1)}$$

If the initial point x_1 is not too far from r, the sequence (x_n) converges very rapidly to r.

Thm. Let $I = [a, b]$ and

1. let $f : I \to \mathbb{R}$ be twice differentiable, and

2. Suppose that $f(a) f(b) < 0$.

3. Assume there are constants m and M such that

$$|f'(x)| \geq m > 0$$ and

$$|f''(x)| \leq M.$$ Let $K = \frac{M}{2m}$.

Then there is a subinterval I^* containing a zero r of f such that for any $x_1 \in I^*$, the sequence (x_n) defined by

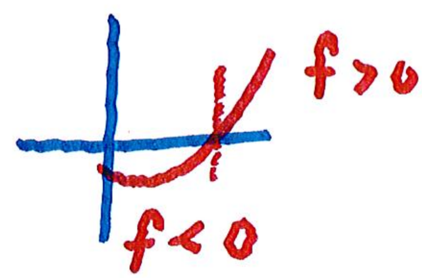
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \in N$$

belongs to I^* and converges to r. In fact

$$|x_{n+1} - r| \leq K |x_n - r|^2 .$$

Pf. $f(a)f(b) < 0$, so $f(a)$ and $f(b)$ have opposite signs.

Also, since $f' \neq 0$ on I, so there is a single point r where $f(r) = 0$



Let $x' \in I$ be arbitrary.

Using Taylor's Thm. there is a point c' between x' and r such that

$$0 = f(r) = f(x') + f'(x')(r - x') + \frac{1}{2} f''(c')(r - x')^2,$$

which implies

$$-f(x') = f'(x')(r - x') + \frac{1}{2} f''(c)(r - x')^2$$

If x'' is the number defined by Newton's Procedure:

$$x'' = x' - \frac{f(x')}{f'(x')},$$

then a calculation implies

$$-\frac{f(x')}{f'(x')} = r - x' + \frac{1}{2}\frac{f''(c')(r-x')^2}{f'(x')}$$

or

$$x'' = x' - \frac{f(x')}{f'(x')} = r - x' + \frac{1}{2}\frac{f''(c')}{f'(x')}(r-x')^2$$

which implies

$$x'' - r = \frac{1}{2}\frac{f''(c')}{f'(x')}(r-x')^2.$$

Using the bounds for $|f'|$ and $|f''|$ and setting $K = \frac{M}{2m}$, we obtain

$$|x'' - r| \le K\,|x' - r|^2 \qquad (3)$$

We now choose $\delta > 0$ so small so that $\delta < \frac{1}{K}$, and that the interval $I^* = [r - \delta, r + \delta]$ is contained in I. If $x_n \in I^*$, then $|x_n - r| \le \delta$

Hence (1) implies

$$|x_{n+1} - r| \le K\,|x_n - r|^2$$

$$\le K\delta^2 < \delta$$

Hence $x_n \in I^*$ implies that

$x_{n+1} \in I^*$ for all $n \in \mathbb{N}$.

By induction, one can

show that $|x_{n+1} - r| < (K\delta)^n |x_1 - r|$

for all n; Thus $\lim x_n = r$.

It still remains to show that

x_n converges to r.

We prove by induction that

(4) $|x_{n+1} - r| < (K\delta)^n |x_1 - r|$ for all $n = 0, 1, \ldots$

(Recall that $K\delta < 1$).

When $n = 0$, the statement is obvious. Suppose that we've proved that

$$|x_N - r| < (K\delta)^{N-1} |x_1 - r| \text{ for some } N \geq 1.$$

By (3),

$$|x_{N+1} - r| \le K|x_N - r||x_N - r|$$

$$\le K\delta\ |x_N - r|$$

(since we showed above that

$|x_n - r| \le \delta$)

$$\le K\delta\ (K\delta)^{N-1} |x_1 - r|$$

$$= (K\delta)^N |x_1 - r|,$$

where the last statement followed by the inductive hypothesis

By the principle of Mathematical Induction, it follows that

$$|x_{N+1} - r| \le (K\delta)^N |x_1 - r|$$

for all $N \ge 0$.

Since $K\delta < 1$, $\lim\limits_{N\to\infty} |x_{N+1} - r| = 0$