Let's try a different proof

of Taylor's Theorem:

Suppose that

(1) f is continuous in the closed interval determined by a and x;

(2) $f^{(n)}(a)$ exists;

(3) $f^{(n+1)}$ exists in the interior of I.

Then $f(x) = P_n(x) + R_n(x)$,

where $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}$ and

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

where c is some point

in the interior of I.

Proof To simplify the notation, we shall abbreviate $R_n(t)$ to $R(t)$.

The essential facts about

$R(t)$ are:

(a) $R(a) = R'(a) = \dots = R^{(n)}(a)$

(b) $R^{(n+1)}(t) \equiv f^{(n+1)}(t)$.

The latter fact comes from

the fact that

$$R(t) = f(t) - P_n(t)$$

and that $P_n(t)$ is a polynomial

of degree at most n.

We introduce a function φ as follows:

$$\varphi(t) = R(t) - K(t-a)^{n+1},$$

where K is a constant which we choose so that $\varphi(x) = 0$. Thus

$$K = \frac{R(x)}{(x-a)^{n+1}}.$$

(Keep in mind that x is fixed

throughout the entire argument.) The function φ has these properties:

a') $\varphi(x) = \varphi(a) = \varphi'(a) = \dots = \varphi^{(n)}(a) = 0$;

b') $\varphi^{(n+1)}(t) - (n+1)!\,K$.

We now apply Rolle's theorem to φ and its derivatives.

Since $\varphi(x) = \varphi(a) = 0$, there exists by Rolle's Theorem a number c_1 between a and x such that $\varphi'(c_1) = 0$. If $n = 0$, we are done. Otherwise, since $\varphi'(a) = \varphi'(c_1) = 0$, there must exist a c_2 between a and c_1 (and hence between a and x) such that $\varphi''(c_2) = 0$.

By applying Rolle's Theorem n times, we conclude that there is a number c between a and x such that $\varphi^{(n+1)}(c) = 0$. Thus,

$$0 = \varphi^{(n+1)}(c) = f^{(n+1)}(c) - (n+1)!\,K,$$

so that

$$R_n(x) = K(x-a)^{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

This complete's the proof of Taylor's Theorem with Lagrange's Remainder.

Corollary. Let I be an interval containg a and suppose that

$$|f^{(n+1)}(x)| \leq M \quad \text{for all } x \in I,$$

where M is a constant. Then

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

Ex. Consider the function $f(x) = e^x$. Note that for any fixed $d > 0$,

$$\sup\{ f^{(n)}(x) \; ; \; |x| \leq d \} = e^d.$$

Thus $M = e^d$. If we write

$$P_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!}.$$

By the error estimate,

$$\left| f(x) - \sum_{n=0}^{N} \frac{x^n}{n!} \right|$$

$$\leq \frac{e^d \, |x|^{N+1}}{(N+1)!} \to 0 \text{ as } N \to \infty.$$

We conclude that

$$\lim_{N\to\infty} |e^x - P_N(x)| = 0,$$

for all x with $|x| \leq e^d$.

Hence $e^x = \sum_{n=0}^{N} \frac{x^n}{n!}$.