Def'n. Let $(b_n)$ be a sequence of numbers that is bounded above. We define

$$v_n = \sup\{ b_k : k \geq n \}.$$

Note that $(v_n)$ is decreasing, i.e., $v_{n+1} \leq v_n$ for all $n$.

We define $v$ = limit of $(v_n)$.

Note that $v$ is either finite or equal to $-\infty$.

We write $v =$ limit superior of $(b_n)$, or more commonly,

$$v = \limsup (b_n).$$

If $v' > v$, then there is a number $N > 0$ so that

$$v \le v_N < v'.$$

Hence, $v \le \sup\{b_n ; n \ge N\} < v'$.

If $v = -\infty$, then one can show that $(v_n)$ decreases to $-\infty$.

Def'n. Let $\sum_n a_n x^n$ be a power series. If the sequence $|a_n|^{1/n}$ is bounded, we set

$$\rho = \lim(|a_n|^{1/n}).$$

If the sequence $(b_n)$ is

not bounded above, then we set $\rho = +\infty$.

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Definition. We define the radius of convergence by

$$R = \begin{cases} 0 & \text{if } \rho = +\infty. \\ 1/\rho & \text{if } 0 < \rho < \infty \\ +\infty & \text{if } \rho = 0 \end{cases}$$

Theorem (Cauchy-Hadamard)

If R is the radius of

convergence of the power series, then the series is absolutely convergent if $|x| < R$ and is divergent if $|x| > R$. When $R = +\infty$, the series converges absolutely for all $x$, and when $R = 0$, the series convergence only when $x = 0$.

Proof: Suppose first that $0 < |x| < R$. Then there is a positive number $c < 1$ such that $|x| < cR$. Therefore

$$\rho < \frac{c}{|x|},$$

and so, if $n$ is suffiently large, then $|a_n|^{1/n} \leq \frac{c}{|x|}$.

This is equivalent to

$|a_n x^n| \leq c^n$ for $n$ sufficiently large.

Since $\sum c^n$ is a geometric series, $\sum |a_n x^n|$ converges absolutely.

If $|x| > R = \frac{1}{\rho}$,

then there are infinitely many $n \in \mathbb{N}$ for which

$$|a_n|^{1/n} > \frac{1}{|x|}.$$

Then there are equivalently many $n$ such that $|a_n x^n| > 1$.

Thus, the series does not

converge.

If $\rho = 0$, $(R = \infty)$,

then for any small number $c$,

$|a_n|^{1/n} \leq c$, if $n$ is suff. large

This means that

$|a_n| \leq c^n$, for $n \geq N$.

Hence, if $x$ is chosen with

$|x| < \frac{1}{d}$, where $\frac{c}{d} < 1$.

then $|a_n x^n| < c^n \cdot \left(\frac{1}{d}\right)^n$.

Since $\frac{c}{d} < 1$, it follows

that $\sum |a_n x^n|$ converges

for all $x$ with $|x| < \frac{1}{d}$.

or equivalently $|x| < \frac{1}{c}$.

Since $c$ is arbitrarily small,

$\sum |a_n x^n|$ converges for all $x$.

The Cauchy-Hadamard Theorem

says that the Radius of

Convergence is $\dfrac{1}{\limsup |a_n|^{1/n}} = R.$

Now we give a theorem

that shows that we can

differentiate power series.