Today we show that if

$$\sum_{k=0}^{\infty} a_k x^k$$ converges to $S(x)$

when $|x| < R$, then $S(x)$

is differentiable when $|x| < R$

and $S'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$.

Thus we compute the series

of $S'(x)$ by differentiating

term by term.

But first:

Thm. 1. Suppose that $\{S_n\}$ is a sequence of functions each of which is continuously differentiable on an interval $I: \{a \leq x \leq b\}$ Suppose further that $\{S_n\}$ converges at one point x_0 in I and that $\{S_n'\}$ converges uniformly in I.

Then $\{S_n\}$ converges

uniformly in I to a

function S, and that

$$S' = \lim_{n\to\infty} S_n'.$$

Proof: By the Fundamental

Theorem of Calculus,

for any $x \in I$, we have

$$S_n(x) = \int_{x_0}^{x} S_n'(t)\,dt - S_n(x_0).$$

Thus

$$S_n(x) - S_m(x) = \int_{x_0}^{x} [S_n'(t) - S_m'(t)]\, dt + [S_n(x_0) - S_m(x_0)]$$

$$|S_n(x) - S_m(x)| \le \int_{x_0}^{x} |S_n'(t) - S_m'(t)|\, dt + |S_n(x_0) - S_m(x_0)|$$

Let $\varepsilon > 0$, then the Cauchy Criterion implies there is an integer $N(\varepsilon)$ such that if $m, n > N$

then $|S_n'(t) - S_m'(t)| < \varepsilon$

and $|S_n(x_0) - S_m(x_0)| < \varepsilon$.

Thus

$$|S_n(x) - S_m(x)| \leq \varepsilon(x - x_0) + \varepsilon$$
$$\leq \varepsilon(b - a + 1).$$

and so, $\{S_n\}(x)$ converges uniformly to a number $S(x)$, for each $x \in I$. We denote the limit of S_n' by σ. It remains to show that $S' = \sigma$. We see that

$$S_n(x) - S_n(x_0) = \int_{x_0}^{x} S_n'(t)\,dt$$

After taking limits on both sides, we get

$$S(x) - S(X_0) = \int_{x_0}^{x} \sigma(t)\,dt.$$

By the Fundamental Theorem of Calculus,

$$S'(x) = \sigma(x).$$

Theorem. Let $\sum_{n=0}^{n=\infty} a_n x^n$ have a nonzero radius of convergence R. Denote its sum by $f(x)$. Then f is differentiable for $\{-R < x < R\}$ and

$$f'(x) = \sum_{n=0}^{\infty} a_n \frac{d}{dx}(x^n)$$

$$= \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Proof. Let x_0 be any point in $(-R, R)$. Then there is an $h > 0$, so that the interval $J = \{x_0 - h \le x \le x_0 + h\}$ is in $(-R, R)$. Hence both

$$\sum a_n x^n \quad \text{and} \quad \sum n a_n x^{n-1}$$

are uniformly convergent in J, so that by Theorem 1, the termwise differentiable is justified in $\mathcal{J}$, in particular at x_0. Since x_0 is an arbitrary point in $\{-R < x < R\}$, the differentiable at all points in this interval.