Sets can be arbitrarily large: For any set S, let

$\wp(S)$ be the set of all subsets of S.

Cantor's Thm:
There does NOT exist a map $\phi : S \longrightarrow \wp(S)$ that is onto.

Proof. Suppose

$$\varphi : S \longrightarrow \wp(S)$$

is a surjection.

Since $\varphi(x)$ is a subset of S. Either x belongs to $\varphi(x)$ or it does not belong to $\varphi(x)$. We let

$$D = \left\{ x \in S : x \notin \varphi(x) \right\}$$

Since φ is a surjection,

there exists $x_0 \in S$
such that $\varphi(x_0) = D$.

There are 2 cases:

1. Suppose $x_0 \in D$.

Then $x_0 \in \varphi(x_0)$.

By definition of D,

$x_0 \notin D$. Contradiction

2. Suppose $x_0 \notin D$.

Then $x_0 \notin \varphi(x_0)$.

By definition of D,

$x_0 \in D$. Contradiction.

Ex. Suppose $S = \{a, b, c\}$

$\wp(S) = \{ \phi, \{a\}, \{b\}, \{c\},$

$\{a, b\}, \{a, c\}, \{b, c\}$

and $\{a, b, c\} \}$

S has 3 elements,

$\mathcal{P}(S)$ has 8 elements.

There does not exist

a surjection from

S onto $\mathcal{P}(S)$.

2.1 Algebraic and Order Properties of $\mathbb{R}$.

On $\mathbb{R}$, there are two operations; addition + multiplication. They satisfy:

(A1) $a+b=b+a$, (commutative addition)

(A2) $(a+b)+c=a+(b+c)$

(associative addition)

(A3) There is an element 0 in $\mathbb{R}$ so $a+0=a$

(0-element exists)

(A4) For each a in $\mathbb{R}$, there is an element $-a$ in $\mathbb{R}$ so that

$a+(-a) = 0$ and $(-a)+a = 0$

(negative element)

(M1) $a \cdot b = b \cdot a$ (commutative multiplication)

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(associative multiplication)

(M3) There is an element 1 in $\mathbb{R}$

so that $a \cdot 1 = 1 \cdot a = a$

(unit element exists)

(M4). For each $a \neq 0$ in $\mathbb{R}$,

there exists an element

$1/a$ such that

$a \cdot \left(\frac{1}{a}\right) = 1$ and

$(1/a) \cdot a = 1$

(existence of reciprocal)

(D) $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$

and

$(b+c) \cdot a = (b \cdot a) + (c \cdot a)$

(distributive property)

In a word, $\mathbb{R}$ is a field

By applying some of the above properties, one can show that the

(1) zero element 0, the

(2) unit element 1, and

(3) the reciprical $\frac{1}{a}$ are

all unique.

For example, suppose $a \neq 0$

and $a \cdot b = 1$. Then

$$b \underset{(M_3)}{=} 1 \cdot b \underset{(M4)}{=} \left(\left(\frac{1}{a}\right) \cdot a \right) \cdot b$$

$$\underset{(M2)}{=} \left(\frac{1}{a}\right) \cdot (a \cdot b) \underset{(D)}{=} \left(\frac{1}{a}\right) \cdot 1 \underset{(M3)}{=} \frac{1}{a}.$$

This proves (3)

Also, if $a \in \mathbb{R}$, then $a \cdot 0 = 0$

In fact,

$$a + a\cdot 0 \underset{\text{by }(M_3)}{=} a \cdot 1 + a \cdot 0 \underset{\text{by }(D)}{=} a \cdot (1+0)$$

$$\underset{\text{by }(A_3)}{=} a \cdot 1 \underset{\text{by }(M_3)}{=} a$$

Adding $(-a)$ to both sides, we get

$a \cdot 0 = 0.$

Also, $0 = (-1)(-1+1) = (-1)(-1) + (-1).$

Adding 1 to both sides, we get

$(-1)(-1) = 1$

We define subtraction by

$$a - b = a + (-b)$$

and also we write

$$ab = a \cdot b,$$

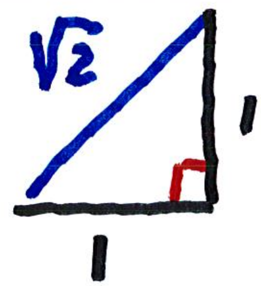
and $a^2 = aa$ and

$a^3 = a^2 a$ and

$a^{n+1} = a^n a$, etc.

$\mathbb{Q}$, $\mathbb{R}$ are both fields.

Thm. There does not exist a rational number r such that $r^2 = 2$



Suppose by contradiction that $r = p/q$. Then

$$r^2 = (p/q)^2 = 2 \rightarrow p^2 = 2q^2.$$

We can assume that

p and q have no common

factor. Then at <u>most one of p and q</u> is even.

Since $p^2 = 2q^2$, we see that p^2 is even. This implies that <u>p is also even</u> (because if $p = 2n+1$ is odd, then $p^2 = 4n^2 + 4n + 1$ is also odd.)

Hence we can write <u>$p = 2m$</u>,

so that

$$p^2 = 4m^2 = 2q^2.$$

Dividing by 2,

$$2m^2 = q^2.$$

Hence q^2 must be even,

which implies <u>q is even.</u>

This shows that both

p and q are even, which

is a contradiction.

It follows that

$\mathbb{R}$ must include numbers

that are irrational

(i.e., not rational).

For this purpose we need to

study Order Properties.

i.e., $<$ and $>$.

Order Properties of $\mathbb{R}$

There is a nonempty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of positive real numbers such that

(i) If $a, b \in \mathbb{P}$, then $a+b \in \mathbb{P}$

(ii) If $a, b \in \mathbb{P}$, then $ab \in \mathbb{P}$

(iii) If $a \in \mathbb{P}$, then exactly one of the following holds:

$a \in \mathbb{P}$, $\quad a = 0$, $\quad (-a) \in \mathbb{P}$

Trichotomy Property

If $-a \in \mathbb{P}$, we say a is negative, and we write $a < 0$ or $0 > a$.

If $a \in \mathbb{P}$, we write $a > 0$ or $0 < a$

If $a \in \mathbb{P} \cup \{0\}$, we write $a \geq 0$.

If $-a \in \mathbb{P} \cup \{0\}$, then we write $a \leq 0$.

If (i)–(iii) hold, then we say $\mathbb{R}$ is an ordered field.

Applying the Trichotomy Property to $a-b$, we get

If $a-b \in \mathbb{P}$, i.e. $a > b$.

If $-(a-b) \in \mathbb{P}$, then $(b-a) \in \mathbb{P}$

$\Rightarrow b > a$

If $a-b=0$, then $a=b$

Here are the Rules for Inequalities :

Thm. 2.1.7 Let $a, b, c \in \mathbb{R}$.

(a) If $a > b$ and $b > c$, then

$a > c$

(b) If $a > b$, then $a + c > b + c$

(c) If $a > b$ and $c > 0$, then

$ca > cb$

If $a > b$ and $c < 0$, then

$ac < ab$

Proof of (a): $a - b > 0$, $b - c > 0$

then $(a - b) + (b - c) > 0$

or $a - c > 0 \rightarrow a > c$

(b) If $a - b > 0$, then

$$(a+c) - (b+c) = a - b > 0$$

$$\rightarrow a + c > b + c$$

(c) If $a > b$ and $c > 0$, then

$$ca - cb = c(a-b) > 0.$$

$$\rightarrow ca > cb$$

If $c < 0$, then $-c > 0$. Hence

$$c(b-a) = -c(a-b) > 0$$

$$\rightarrow cb - ca > 0 \rightarrow cb > ca.$$

The Order Properties in 2.1.5 and 2.1.6 lead to 2.1.10 and 2.1.11, which are useful for solving inequalities:

1. Suppose that $ab > 0$. If $a > 0$, then $b > 0$.
2. If $ab > 0$ and $a < 0$, then $b < 0$
3. If $ab < 0$ and $a > 0$, then $b < 0$
4. If $ab < 0$ and $a < 0$, then $b > 0$

Finally, we need to prove several facts:

Thm 2.1.8

(a) if $a \in \mathbb{R}$ and $a \neq 0$, then

$a^2 > 0$

(b) if $n \in N$, then $n > 0$

Since $1 = 1^2$, (a) $\Rightarrow$ $1 > 0$

(c) If $n \in N$, then $n > 0$.

Apply (b) and (i) from Order Properties. Use Math. Ind.

(d) If $a > 0$, then $a^{-1} > 0$.

(e) If $0 < a < b$, then

$$a^{-1} > b^{-1}.$$

Pf. of (d). Suppose that

$a^{-1} < 0$. Then

$$1 = aa^{-1} < a \cdot 0 = 0.$$

This contradiction shows

$a^{-1} > 0$.

Pf. of (e). If $0 < a < b$,

then $a^{-1} - b^{-1} = (ab)^{-1}(b-a) > 0$

Since $(ab)^{-1} > 0$ and $b-a > 0$,

we get $a^{-1} > b^{-1}$.

Ex. Find all real numbers x such that $3x+4 \le 12$.

Justify each step.

$$3x+4 \le 12 \iff 3x \le 8 \iff x \le \frac{8}{3}$$

By (b) of 2.1.7

By (c) of 2.1.7

Ex. Solve $x^2-4x-5<0$.

$$x^2-4x-5 = (x-5)(x+1) < 0$$

$\iff$ $\underbrace{\text{If } x-5>0 \text{, then } x+1<0}_{\text{No solution.}}$

By Property (3) above

Or, by Property 4, if

$x-5 < 0$, then $x+1 > 0$.

$\therefore$ Solution is $-1 < x < 5$.

~~Finally, we have~~

~~Thm. 2.1.8:~~

~~(a) if $a \in \mathbb{R}$ and $a \neq 0$,~~

~~then $a^2 > 0$.~~

~~(b) $1 > 0$. Since $1 = 1^2$~~

~~this follows from (a)~~

We will define $\mathbb{R}$ as the set of infinite decimal expansions:

$$x = \pm B . b_1 b_2 \ldots ,$$

where B is a non-negative integer and b_j is the coefficient of 10^{-j} and

$$0 \leq b_j \leq 9$$

For example,

$$\pi = 3.14159265\ldots$$

$$e = 2.718281828\,45\ldots$$

$$\sqrt{2} = 1.4142135623\ldots$$

It turns out that rational numbers are those decimal expansions that are periodic.

Express $x = 45.2343434\ldots$

Multiply by 10.

$$10x = 452.3434\ldots$$

Multiply $10x$ by 100

$$1000x = 45234.3434\ldots$$

Subtract:

$$990x = (45234 - 452)$$

$$x = \frac{44782}{990}.$$