Absolute Value 2.2.

We can define $|a|$ as follows:

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

We'll need these identities:

(i) $|-a| = |a|$

(ii) $|ab| = |a||b|$

(iii) $|a|^2 = a^2$

(iv) $-|a| \le a \le |a|$

These four identities can all be proved by considering the cases when $a>0$, $a<0$ and $a=0$, and then applying the definition of $|a|$.

For example, to prove (iv), suppose that $a \geq 0$. Then

$$-|a| = -a \leq 0 \leq a = |a|$$

Then, if $a < 0$,

$$-|a| = -(-a) = a < 0 \le |a|.$$

This proves (iv) in both cases.

The following inequality is very useful.

Triangle Inequality.

If $a, b \in \mathbb{R}$, then

$$|a+b| \le |a| + |b|.$$

Pf. Suppose first that $a+b \ge 0$

$\rightarrow |a+b| = a+b \le |a| + |b|$

using (iv)

Now suppose that $a + b < 0$.

Then

$|a+b| = -(a+b)$

$= -a - b \leq |a| + |b|$,

which proves the Triangle Property, again by (iv).

Another version is the

Backwards Triangle Property

$$|a-b| \geq |a| - |b|.$$

Pf.

$$|a| = |(a-b)+b|$$

$$\leq |a-b| + |b|$$

$$\Rightarrow |a-b| \geq |a| - |b|$$

One more identity

Suppose $c \geq 0$. Then

(1) $|a| \leq c$ if and only if

$$-c \leq a \leq c.$$

Proof:

Case 1: Assume $a \geq 0$.

$|a| \leq c \rightarrow a \leq c$

$\rightarrow -c \leq 0 \leq a \leq c$

Case 2: Assume $a < 0$.

$-a = |a| \le c$

$\rightarrow -c \le a < 0 \le c$

Thus, in both cases, we get the desired inequality.

Now, let's prove the "if" direction. We know

$a \le c$. Also, $-c \le a$

or $-a \le c$

We obtain $|a| \le c$

Thus, we've proved both directions.

Ex. Find the set A of all x such that $|3x+4| < 2$

Set $c = 2$

$\therefore$ Left half is

and $a = 3x+4$.

$|a| < c \rightarrow -c < a < c$

or $-2 < 3x+4 < 2$

$\therefore \quad -6 < 3x < -2$

$$-2 < x < -\frac{2}{3}.$$

Def'n. Let $a \in \mathbb{R}$ and $\varepsilon > 0$.

Then the ε- neighhorhood of a is the set

$$V_\varepsilon(a) = \left\{ x \in \mathbb{R} : |x-a| < \varepsilon \right\}$$

Note that $|x-a| < \varepsilon$

is equivalent to

$$-\varepsilon < x-a < \varepsilon$$

or

$$a-\varepsilon < x < a+\varepsilon.$$

On the real line this is

Thm. Let $a \in \mathbb{R}$. If x belongs to $V_\varepsilon(a)$ for every $\varepsilon > 0$, then $x = a$.

Pf. Suppose $x \neq a$. If we set $\varepsilon = \frac{|x-a|}{2}$ in the definition of $V_\varepsilon(a)$, then

$$|x-a| < \frac{|x-a|}{2}.$$

Dividing by $|x-a|$, we have $1 < \frac{1}{2}$. This contradiction $\rightarrow$ $x = a$.

2.3 We said that $\mathbb{R}$ can be defined as the set of decimal expansions. Thus a real number x can be defined by

$$x = \pm B . b_1 b_2 b_3 \cdots ,$$

where B is a non-neg. integer.

$$e = 2.718281828459045\cdots$$

But there several others. It turns out all such definitions. Most importunately,

$\mathbb{R}$ has the Least Upper Boundary. To explain this, let S be a non-empty subset of $\mathbb{R}$.

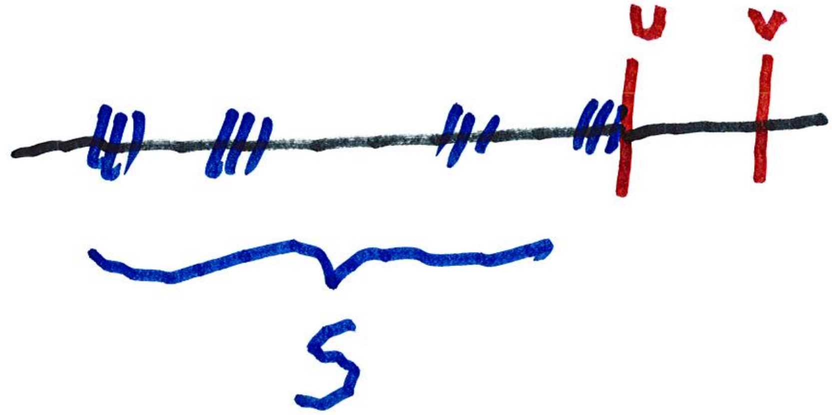
Def'n. We say S is bounded above if there is a number u such that $s \leq u$, for all $s \in S$

We say that the number u is a least upper bound of S

if

(i) u is an upper bound of S, and

(ii) If v is any upper bound of S, then $v \geq u$.



Similarly, we say that the number w is a lower bound of S if $w \le s$ for all $s \in S$.

We say w is a greatest lower bound of S if

(i) w is a lower bound of S

and

(ii) If t is any lower bound. then $t \le w$

We often write

l.u.b. S = supremum S = sup S

g.l.b. S = infimum S = inf S

One can show

Thm. If S is bounded above, (has an upper bound) then S has a least upper bound.

and if S is bounded below.

then S has a greatest lower bound.

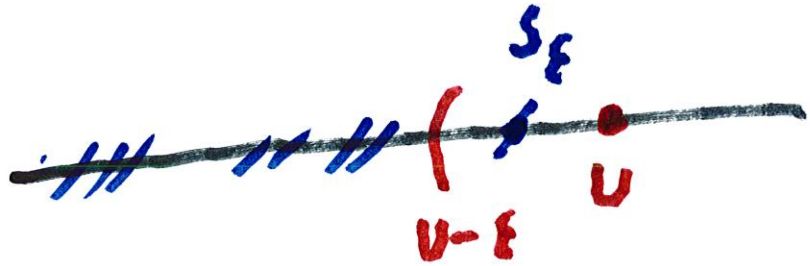
Thm. Consider a set S that is nonempty and bounded above. Let u be the least upper bound.

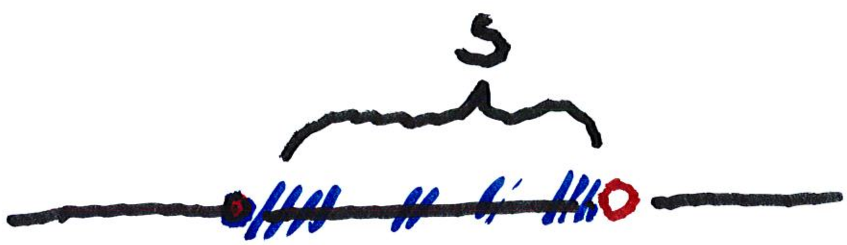
Then, there are 2 cases.

1. $u \in S$. 

or 2. For every $\varepsilon > 0$, there is an element $s_\varepsilon \in S$, such that $u - \varepsilon < s_\varepsilon < u$

Pf. Since $u - \varepsilon$ is <u>not</u> an upper bound, there is an element $s_\varepsilon \in S$, such that $u - \varepsilon < s_\varepsilon < u$.

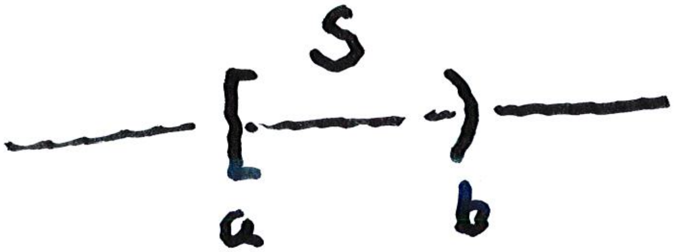




Note that inf S and sup S

do not always belong to S.

Ex. Let $S = [a, b)$, where $a < b$.



Find inf S and sup S

$x \in [a,b) \to a \le x < b.$ (*)

$\therefore$ b is an upper bound of $[a, b)$

Let v be any upper bound of S. Then $v \geq s$ for all $s \in S$.

Suppose that $v \leq a$. Then $s = \dfrac{a+b}{2}$ is in S, but $v < s$, which would show v is <u>not</u> an upper bound of S.

Suppose that $a < v < b$. Then $s = \dfrac{v+b}{2}$ is in S, but $v < s$, which shows that

v is not an upper bound of S.

It follows that $v \geq b$, which shows that b is the least upper bound of S

Now note that the first inequality in $(*)$ shows that a is a lower bound of S.

$\therefore$ $(1')$ holds

Since $s = a$ is in S, it follows that any lower bound t of S must satisfy $t \leq a$. Hence,

$a =$ g.l.b of S.

or $\inf S = a$