2.3 Least Upper Bounds.

Let S be a nonempty subset of $\mathbb{R}$ such that there is a number M such that $x \leq M$, for all $x \in S$, then we say a number u is an upper bound of S if

$$s \leq u, \quad \text{for all } s \in S$$

Definition. We say u is a least upper bound of S

if (1) u is an upper bound,

and (2) If v is any upper bound of S, then $u \le v$.

Essentially, u is the "maximum" of S

Similarly, we say a number

w is a lower bound of S

if $w \le s$, for all $s \in S$.

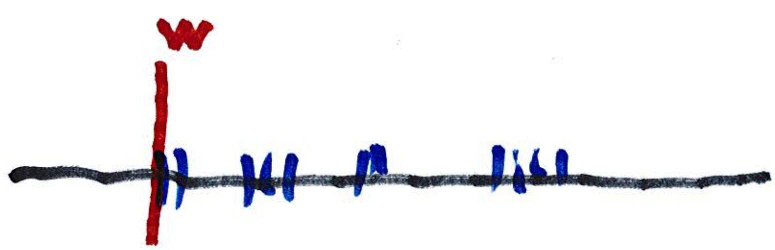
We say w is a greatest

lower bound if

(1') w is a lower bound of S

and (2') If t is any lower

bound of S, then $\quad \ge \quad$.



Ex. Let $S = \{s : a \leq s < b\}$

Note that $s < b$ for all $s \in S$

$\therefore$ b is an upper bound of S.

Suppose that v is any upper bound of S that satisfies $\underline{v < b}$. Then let s satisfy

$s \in (a, b) \cap (v, b)$.

$s \in (a, b) \rightarrow s \in S$

$s \in (v, b) \rightarrow s > v$.

It follows that v is not an upper bound of S. Hence, if v is any upper bound of S, then it must be that $v \geq b$.

(i) Note that $a \leq s$ for all $s \in S$. Hence a is a lower bound of S.

(ii) If t is a lower bound of S, then $t \leq s$, for all $s \in S$. which implies that $t \leq a$. (by setting $s = a$)

Thm. Let S be a subset of $\mathbb{R}$ and suppose that u is an upper bound. Then the following statements are equivalent.

(1) If v is any upper bound of S, then $u \le v$.

(2) If $z < u$, then z is not an upper bound of S.

(3) If $z < u$, then there exists $s_z \in S$ such that $z < s_z$.

(4) If $\varepsilon > 0$, then there exists $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

(1) $\to$ (2). If $z < u$ and if z is not an upper bound, then $v = z$ is not an upper bound of S.

(2) → (3) If there is no $s_z \in S$ with $z < s_z$, then $z \geq s$ for all $s \in S$. Hence z is an upper bound, which contradicts (2). Hench (3) is proved.

(3) → (4). Replace z by $u - \varepsilon$. Then if $\varepsilon > 0$, there $s_\varepsilon \in S$ so that $u - \varepsilon < s_\varepsilon$

(4) → (1). If u is an upper bound and if $v < u$, then we put $\varepsilon = u - v$. Then $\varepsilon > 0$, so there exists an $s_\varepsilon \in S$ such that $v = u - \varepsilon < s_\varepsilon$.

Therefore, v is not an upper bound of S and

2.4. Applications of Least Upper Bound Property.

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence.

1. We say $\{x_n\}$ is increasing if $x_{n+1} \geq x_n$, for all $n = 1, 2, \ldots$

2. We say $\lim_{n \to \infty} x_n = \tilde{x}$ if for all $\varepsilon > 0$, there is an

integer $N_\varepsilon > 0$ so that if

$n \geq N_\varepsilon$, then

$$|x_n - \tilde{x}| < \varepsilon, \quad \text{for all } n \geq N_\varepsilon.$$

Thm. Suppose $\{x_n\}$ is an

increasing sequence such that

$x_n \leq M$, for all $n = 1, 2, \dots$.

Then there is a number

$\tilde{x} \leq M$, such that

$$\lim_{n\to\infty} x_n = \tilde{x}.$$

Pf. Let $S = \{x_n ; n=1,2,\ldots\}$

and let $\tilde{x} = \text{l.u.b } S$.

Choose $\varepsilon > 0$. Then

there is an integer $N_\varepsilon > 0$

so that $x_{N_\varepsilon} > \tilde{x} - \varepsilon$.

Since $\{x_n\}$ is increasing,

if $n \geq N_\varepsilon$, then

$$\tilde{x} - \varepsilon < x_{N_\varepsilon} \leq x_n \leq \tilde{x}.$$

The last inequality follows from the fact that

$$x_n \leq \tilde{x} = \text{l.u.b. } S.$$

Hence $\tilde{x} - \varepsilon < x_n \leq \tilde{x} < \tilde{x} + \varepsilon$

i.e., $-\varepsilon < x_n - \tilde{x} < \varepsilon$

for $n \geq N_\varepsilon$.

$$\therefore \lim_{n \to \infty} x_n = \tilde{x}.$$