Review Problems, Test 1.

p. 15 # 2. Prove that

$$1^3 + 2^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

It's obvious if $n=1$

Suppose it's true for n.

$$1 + 2^3 + \dots + n^3 + (n+1)^3$$

$$= \frac{n^2(n+1)^2}{4} + (n+1)^3$$

$$= (n+1)^2 \left[\frac{n^2}{4} + (n+1) \right]$$

$$= (n+1)^2 \left[\frac{n^2 + 4n + 4}{4} \right]$$

$$= \left(\frac{(n+1)(n+2)}{2} \right)^2 \quad \checkmark$$

p. 37 A nonempty set S in $\mathbb{R}$ is bounded above if there is a number u so $x \le u$, for all $x \in S$.

If S is bounded above, then u is a least upper bound if

(1) u is an upper bound, and

(2) if v is any upper bound, then

$$u \leq v$$

Observe that if $\varepsilon > 0$, then

there is an $s_\varepsilon \in S$ such that

$$u - \varepsilon < s_\varepsilon \leq u$$

p. 48

2.5.2 Nested Intervals Property

If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$ is a nested sequence of closed intervals, then there is a number $\xi \in \mathbb{R}$ so that

$\xi \in I_n$ for all $n \in \mathbb{N}$.

Pf. Since the intervals are nested, we have $I_n \subseteq I_1$. Hence

$a_n \le b_1$ for all $n \in N$. Hence

~~If~~ $S = \{a_n ; n \in N\}$ has a

supremum ξ. Clearly

$a_n \le \xi$ for all $n \in N$.

We want to show that

$\xi \le b_n$ for all n. To do this,

we show b_n is an upper bound

of S. There are 2 cases.

(ii) If $n \leq k$, then since

$I_n \supseteq I_k$, we have $a_k \leq b_k \leq b_n$.

(iii) If $k < n$, then since

$I_k \supseteq I_n$, we have

$a_k \leq a_n \leq b_n$.

Thus $a_k \leq b_n$ for all k, so that

b_n is an upper bound of S.

Hence, $\xi \leq b_n$ for each $n \in \mathbb{N}$.

Since $a_n \leq \xi \leq b_n$ for all n,

we have $\xi \in I_n$ for all n.

Suppose $\lim_{n \to \infty} (b_n - a_n)$. Then

the number ξ is unique.

Pf. Set $\eta = \inf S'$, where

$S' = \{ b_n ; n \in \mathbb{N} \}$. By following

the above proof, we obtain

$\eta = \inf S'$.

One can show that there is a_k arbitrarily close to ξ and a b_k arbitrarily close to η.

Hence $\lim_{n\to\infty} (b_n - a_n) = \eta - \xi$.

p. 56 Uniqueness of Limits

Pr sequence can only have at most one limit.

Pf. Suppose that $x' = \lim x_n$ and $x'' = \lim x_n$, where $x' \neq x''$. Then there is a K' so that if $n \geq K'$, then $|x_n - x'| < \frac{\varepsilon}{2}$, where $\varepsilon = |x' - x''|$

Similarly, there is K'' so that if $n \geq K''$, then $|x_n - x''| < \frac{\varepsilon}{2}$.

Thus, if $n > \operatorname{Max}\{K', K''\}$,

then

$$|x' - x''| = |(x' - x_n) + x_n - x''|$$

$$\leq |x' - x_n| + |x'' - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

$$= |x' - x''|.$$

This contradiction shows that

$x' - x'' = 0.$

p. 63. A convergent sequence of real numbers is bounded.

Pf. Let $\varepsilon = 1$. Then there is a $K > 0$ so that if $n > K$, then

$|x_n - L| < 1$. Hence, for such n,

$$|x_n| \le |x_n - L| + |L| \le 1 + |L| = M$$

Therefore, if we set

$$M_0 = \operatorname{Max}\{|x_1|, \dots, |x_K|, 1 + |L|\},$$

then $|x_n| \le M_0$ for all n.

p. 68 Continuity of $\sqrt{x}$.

State and prove.

Let $X = (x_n)$ be any sequence of real numbers that converges and suppose that $x_n \geq 0$. Then

$\lim(\sqrt{x_n}) = \sqrt{x}$.

p. 71. Monotone Convergence Thm

A monotone sequence of real numbers convergent if and only if

Also,

(a) if (x_n) is bounded and increasing, then

$$\lim (x_n) = \sup \{x_n : n \in N\}$$

(b) if (x_n) is bounded and decreasing, then

$$\lim (x_n) = \inf \{x_n : n \in N\}$$

Pf. 71.

We prove (a).

Let $x = \sup\{x_n ; n \in \mathbb{N}\}$.

Choose $\varepsilon > 0$. Then there is x_N so that $x - \varepsilon < x_N$.

Since (x_n) is decreasing, it follows that

$$x - \varepsilon < x_N \leq x_n \leq x,$$

where the last inequality follows since x is an upper bound.

Hence,

$$x - \varepsilon < x_n \le x < x + \varepsilon,$$

for all $n > K$. Hence

$$-\varepsilon < x_n - x < \varepsilon, \text{ for } n \ge K$$

Hence $\lim x_n = x$