5.3.4 Maximum - Minimum Thm.

Let f be continuous on the interval $I = [a,b]$. Then f has an absolute maximum and an absolute minimum on I.

Pf. Let $S = \{ f(x) ; x \in I \}$.

By the Boundedness theorem f has a least upper bound s'.

Clearly $f(x) \leq s'$ for all $x \in I$.

If there is a point $x_0 \in I$ such that $f(x_0) = S'$, then f has an absolute maximum at x_0.

Hence, we can assume that $f(x) < S'$ for all $x \in I$. It follows that the function

$$g(x) = \frac{1}{S' - f(x)}$$ is continuous on I.

Since s' = least upper bound of S, for every $n = 1, 2, \dots,$ there is an $x_n \in I$ such that

$s' - \frac{1}{n} < f(x_n) < s'$. Hence

$s' - f(x_n) < \frac{1}{n}$, which implies that $n < \frac{1}{s' - f(x_n)}$.

But this means that the continuous function g is not bounded. This contradicts the Boundedness Thm.

Mean Value Thm. Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Then there is at least one point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Proof: Let

$$\varphi(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a).$$

Thus φ satisfies the hypotheses of Rolle's Thm. This means there is a point $c \in (a,b)$, such that $\varphi'(c) = 0$, i.e.

$$0 = f'(c) - \frac{f(b)-f(a)}{b-a},$$

which proves the theorem.

(a) To show that f is integrable, we use Integrability Criterion

on . For a given

Since f is continuous on $[a, b]$, it is uniformly continuous. For a given $\varepsilon > 0$, there is a $\delta > 0$ so that if x', x'' are in $[a, b]$ and $|x' - x''| < \delta$, then

$$|f(x') - f(x'')| < \frac{\varepsilon}{2(b-a)} \quad (1)$$

Choose an integer n so that

$\frac{b-a}{n} < \delta$. Then define a

partition $P = \{x_0, x_1, \dots, x_n\}$ so

that $x_0 = a$, $x_n = b$ and

$x_k - x_{k-1} = \frac{(b-a)}{n} < \delta$. If we

set $M_k = \sup\{f(x) : x_{k-1} \le x \le x_k\}$

and $m_k = \inf\{f(x) : x_{k-1} \le x \le x_k\}$

then (1) implies that

$$M_k \leq f(x_k) + \frac{\varepsilon}{2(b-a)} \quad \text{and}$$

$$m_k \geq f(x_k) - \frac{\varepsilon}{2(b-a)},$$

which gives

$$M_k - m_k \leq \frac{\varepsilon}{(b-a)}.$$

If we multiply by $(x_k - x_{k-1})$, and then sum up for $k = 1, \ldots, n$, we obtain

$$\sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) \qquad (2)$$

$$< \sum_{k=1}^{n} \frac{\varepsilon}{(b-a)} (x_k - x_{k-1})$$

$$= \frac{\varepsilon}{(b-a)} (b-a) = \varepsilon.$$

We write

$$U(f, P) = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$$

and $L(f: P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$.

Thus (2) is

$$U(f, P) - L(f, P) < \varepsilon.$$

Thus the Integrability Criterion implies that f is integrable on $[a,b]$

(b). To show that f is continuous, we write

$$f(x)-f(a) = \frac{f(x)-f(a)}{x-a}(x-a).$$

If we take the limit as $x \to a$, we obtain

$$0 = \lim_{x \to a} (f(x) - f(a))$$

$$= f'(a) \cdot 0 = 0.$$

Thus $\lim_{x \to a} f(x) = f(a)$,

which means that f is

continuous.

(c) To show that f is constant

we apply the Mean Value

Theorem.

According to the Mean Value Theorem, for any x in $[a, b]$,

there is a point $c \in (a, x]$ so that $f'(c) = \dfrac{f(x) - f(a)}{x - a}$.

Since $f'(c) = 0$, it follows that $f(x) - f(a) = 0$. Thus,

$\therefore f(x) = f(a)$ for all $x \in [a, b]$.

(d). Thm 7.3.6 states that if

f is continuous on $[a, b]$,

and if $F(x) = \int_a^x f(t)\,dt$, then

if F is antiderivative of f,

i.e., $F'(x) = f(x)$ for $x \in [a, b]$.

If G is also an antiderivative

of f, then $(G - F)'(x) = 0$.

According to the result in (c),

$F(x) - G(x) = C$, for all $x \in [a, b]$,

Hence $\int_a^x f = G(x) + C$.

When $x = a$, we get

$0 = G(a) + C$.

We conclude that

$\int_a^x f = G(x) - G(a)$.

7.3.1 The Fundamental Thm. of Calculus, (First Form) states that if F is continuous on $[a, b]$ and $F'(x) = f(x)$ for all $x \in [a, b]$, then

$$\int_a^b f = F(b) - F(a).$$

Taylor's Theorem with the Remainder states that if $f, f', \ldots, f^{(n)}, f^{(n+1)}$ exist in $[a, b]$, and are all continuous in $[a, b]$, then

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$ where

$$R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)\,(x-a)^n\,dt$$