Exam 2, Fall 2008

Compute $\displaystyle\int_0^1 \frac{dx}{x^2+3x+2}$

EXAM 2 REVIEW

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$\rightarrow \; A(x+2) + B(x+1)$

$$\rightarrow \left.\begin{aligned} A+B &= 0 \\ 2A+B &= 1 \end{aligned}\right\} \rightarrow A=1, \; B=-1$$

$$\rightarrow \int_0^1 \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx = \ln(x+1) - \ln(x+2)\Big|_0^1$$

$$= (\ln 2 - \ln 3) - (\ln 1 - \ln 2)$$

$$= 2\ln 2 - \ln 3$$

2. Find partial fraction decomposition of $\dfrac{2x-4}{x(x^2+4)}$

$$\frac{A}{x} + \frac{Bx+C}{x^2+4} = \frac{2x-4}{x(x^2+4)}$$

$$\rightarrow A(x^2+4) + Bx^2 + Cx = 2x-4$$

$$\rightarrow \begin{aligned} A + B &= 0 \\ C &= 2 \\ 4A &= -4 \end{aligned} \quad \rightarrow \quad \begin{aligned} A &= -1 \\ B &= 1 \\ C &= 2 \end{aligned}$$

$$\rightarrow -\frac{1}{x} + \frac{x+2}{x^2+4}$$

3. Which integral arises when one computes $\int \frac{dx}{\sqrt{x^2+2x+5}}$

$x^2 + 2x + 5 = (x+1)^2 + 4$

$u = x+1 \quad du = dx$

$\to \int \frac{du}{\sqrt{u^2+4}}$ $\quad u^2 = 4\tan^2\theta$

$u = 2\tan\theta$

$du = 2\sec^2\theta \, d\theta$

$$\int \frac{2\sec^2\theta \, d\theta}{\sqrt{4\tan^2\theta + 4}} = \int \sec\theta \, d\theta$$

5. Compute $\displaystyle\int_1^2 \frac{dx}{\sqrt{2-x}}$

Choose t with $1 < t < 2$

$$\int_1^t \frac{dx}{\sqrt{2-x}} = 2\cdot(-1)\cdot(2-x)^{\frac{1}{2}}\Big|_1^t$$

$$= -2\sqrt{2-t} + 2\sqrt{1}$$

$$= 2 - 2\sqrt{2-t}$$

$$\lim_{t\to 2^-}\left(2 - 2\sqrt{2-t}\right) = 2 - 0 = 2$$

6. Which statement about

$$\int_1^\infty \frac{dx}{x^2+x} \text{ is true.}$$

$$\frac{1}{x^2+x} < \frac{1}{x^2} \quad \text{and} \quad \int_1^\infty \frac{dx}{x^2} \text{ converges}$$

$$\therefore \int_1^\infty \frac{dx}{x^2+x} \text{ converges.}$$

A. Integral diverges X

B. Int. converges by comp. with $\int_1^\infty \frac{dx}{x}$

C. Int conv. by comp with $\int_1^\infty \frac{dx}{x^2}$

TRUE

7. Which integral is the area of surface obtained by revolving the curve $y = 2\sqrt{x}$ from $(0,0)$ to $(1,2)$ about the x-axis

$$y = 2\sqrt{x} \qquad \frac{dy}{dx} = 2 \cdot \frac{1}{2} \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{x}}$$

$$SA = \int_0^1 2\pi \cdot 2\sqrt{x} \sqrt{1 + \frac{1}{x}} \, dx$$

$$= \int_0^1 4\pi \sqrt{x+1} \, dx$$

8. Find length of $y = \frac{2}{3}(1+x)^{\frac{3}{2}}$ from $\left(0, \frac{2}{3}\right)$ to $\left(2, 2\sqrt{3}\right)$

$\cdot 3^{3/2}$

$$\frac{dy}{dx} = \frac{2}{3} \cdot \frac{3}{2}(1+x)^{\frac{1}{2}} = \sqrt{1+x}$$

$$\therefore L = \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^2 \sqrt{2+x}\, dx = \frac{2}{3}(2+x)^{\frac{3}{2}} \Big|_0^2$$

$$= \frac{2}{3}\left(4^{3/2} - 2^{3/2}\right)$$

$$= \frac{16}{3} - \frac{4\sqrt{2}}{3}$$

9. Three objects of mass 3 gm, 2 gm and 1 gm are placed at (2,2), (1,1) and (2,4). Where is the centroid of the system

Total mass $= 3+2+1 = 6$

$M_y = 3\cdot 2 + 2\cdot 1 + 1\cdot 2 = 10$

$M_x = 3\cdot 2 + 2\cdot 1 + 1\cdot 4 = 12$

$\therefore\ \bar{x} = \frac{10}{6} \qquad \bar{y} = \frac{12}{6}$

$= \frac{5}{3} \qquad = 2$

10. Planar region is defined by $0 < y < 1+x^2$, $-1 < x < 1$

$$\text{Mass } m = \int_{-1}^{1} 1+x^2 = 2\int_0^1 1+x^2\,dx$$

$$= 2 + \frac{2}{3} = \frac{8}{3}$$

$$M_y = \int_{-1}^{1} x\,(1+x^2)\,dx = 0$$

$$M_x = \frac{1}{2}\int_{-1}^{1} (1+x^2)^2 = \frac{2}{2}\int_0^1 (1+x^2)^2$$

$$= 1 + \frac{2x^3}{3} + \frac{x^5}{5}\Big|_0^1 = \frac{28}{15}$$

$$\therefore \bar{x} = 0, \quad \bar{y} = \frac{28}{15}\cdot\frac{3}{8} = \frac{7}{10}$$

11. Evaluate $\lim_{n \to \infty} \left(n - \frac{n^2 - 2n - 12}{n} \right)$

$$= \lim_{n \to \infty} \left(n - n + 2 - \frac{12}{n} \right)$$

$$= \lim_{n \to \infty} \left(2 - \frac{12}{n} \right) = 2 - 0 = 2$$

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12. $\lim_{p \to \infty} \frac{\ln(4p+5)}{p}$

$$\underset{L}{=} \lim_{p \to \infty} \frac{\frac{1}{4p+5} \cdot 4}{1} = \lim_{p \to \infty} \frac{4}{4p+5} = 0$$

Fall 2010

1. Evaluate $\int_0^{\frac{\pi}{2}} \cos^3 x \sin^2 x \, dx$

$$= \int_0^{\frac{\pi}{2}} (1-\sin^2 x) \sin^2 x \cos x \, dx$$

$u = \sin x \quad du = \cos x \, dx$

$u(0) = 0 \quad u\left(\frac{\pi}{2}\right) = 1$

$$= \int_0^1 (1-u^2) u^2 \, du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} \Big|_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$$

2. $\displaystyle\int_0^{\pi/4} \tan x \sec^4 x$

$$= \int_0^{\frac{\pi}{4}} (\tan^3 x + \tan x) \sec^2 x \, dx$$

$u = \tan x \qquad du = \sec^2 x \, dx \qquad u(0) = 0$

$u\left(\frac{\pi}{4}\right) = 1$

$$= \int_0^1 (u^3 + u) \, du$$

$$= \frac{u^4}{4} + \frac{u^2}{2} \Big|_0^1 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

OR $\displaystyle\int_0^{\pi/4} \sec^3 x \cdot \sec x \tan x \, dx$

$$= \int_1^{\sqrt{2}} u^3 \, du = \frac{u^4}{4} \Big|_1^{\sqrt{2}} = \frac{4}{4} - \frac{1}{4} = \frac{3}{4}$$

3. If we make a trig. substitution

to evaluate $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2-9}}$,

which integral arises?

Want $x^2 = 9\sec^2\theta$ or

$x = 3\sec\theta \rightarrow dx = 3\sec\theta\tan\theta$

$$\rightarrow \int \frac{27\sec^3\theta \cdot 3\sec\theta\tan\theta\,d\theta}{\sqrt{9\sec^2\theta - 9}}$$

$$= 27\int \frac{\sec^4\theta\tan\theta}{\tan\theta}\,d\theta = \int 27\sec^4\theta\,d\theta$$

4. Evaluate $\displaystyle\int_0^{1/\sqrt{2}} \frac{x^2\,dx}{\sqrt{1-x^2}}$

$x = \sin\theta \quad dx = \cos\theta\, d\theta$

$$\to \int_0^{\frac{\pi}{4}} \frac{\sin^2\theta \cdot \cos\theta\, d\theta}{\cos\theta}$$

$$= \int_0^{\frac{\pi}{4}} \sin^2\theta\, d\theta = \int_0^{\frac{\pi}{4}} \frac{1-\cos 2\theta}{2}$$

$$= \frac{\theta}{2} - \frac{\sin 2\theta}{4} \Big|_0^{\pi/4}$$

$$= \frac{\pi}{8} - \frac{1}{4}$$

5. Use Trapezoidal Rule with $n = 4$ to approximate $\int_0^2 (x^2+1)\,dx$

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n) \right]$$

$$\Delta x = \frac{2}{n} = \frac{2}{4} = \frac{1}{2}$$

$f(x_0) =$ $\quad f(x_1) = \left(\frac{1}{4} + 1\right) = \frac{5}{4}$

$f(x_2) = 1 + 1 = 2 \quad f(x_3) = \left(\frac{3}{2}\right)^2 + 1 = \frac{13}{4}$

$f(x_4) = 5.$

$$T_4 = \frac{1}{2}\left\{ 1 + \frac{2 \cdot 5}{4} + 2 \cdot 2 + 2 \cdot \frac{13}{4} + 5 \right\} = \frac{19}{4}$$

6. Let C = curve $y = \sqrt{1+4x}$, $1 \le x \le 5$

Find the area of the surface

obtained by rotating C about the

x-axis.

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{4}{\sqrt{1+4x}}$$

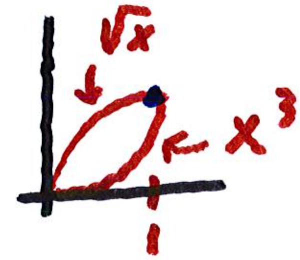
$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{4}{1+4x}$$

Hence $\sqrt{1 + \frac{4}{1+4x}} = \sqrt{\frac{5+4x}{1+4x}}$

$$\rightarrow SA = 2\pi \int_1^5 \sqrt{1+4x} \cdot \sqrt{\frac{5+4x}{1+4x}}$$

$$= 2\pi \int_1^5 \sqrt{5+4x} = 2\pi \cdot \frac{2}{3} \cdot \frac{1}{4} (5+4x)^{\frac{3}{2}} \Big|_1^5$$

$$= \frac{\pi}{3} 25^{3/2} - \frac{\pi}{3} 9^{3/2}$$



7. Let D = region bounded by $y = \sqrt{x}$ and $y = x^3$. If $(\bar{x}, \bar{y})$ is the centroid of D, and A is the area, then find $\bar{x}$

$$\bar{x} = \frac{M_y}{A}, \quad M_y = \int_0^1 x(\sqrt{x} - x^3)\,dx$$

$$= \int_0^1 (x^{3/2} - x^4)\,dx = \frac{2}{5} - \frac{1}{5} = \frac{1}{5}$$

$$\therefore \bar{x} = \frac{1}{5A}$$

Review for Exam 2, Spring 2014

1. Find the Partial Fraction Decomposition of

$$\frac{x^2+1}{x^3(x^2+x+3)^2}$$

x^2+x+3

$= (x+\frac{1}{2})^2 + \frac{11}{4}$

irreducible

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{(x^2+x+3)} + \frac{Fx+G}{(x^2+x+3)^2}$$

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2. Compute $\int \frac{2x^2}{x^2+2x+2}\,dx$

$$x^2 + 2x + 2 \,\overline{\big)\, 2x^2}\quad 2$$

$$2x^2 + 4x + 4$$

$$-4x - 4$$

$$\int = \int 2 + \frac{(-4x-4)}{x^2+2x+2}\, dx$$

$$x^2 + 2x + 2 = (x+1)^2 + 1$$

Set $u = x+1$

$\rightarrow u - 1 = x$

$du = dx$

$$\int \frac{-4x-4}{x^2+2x+2}\, dx = \int \frac{-4(u-1)-4\; du}{u^2+1}$$

$$= \int \frac{-4u \quad du}{u^2+1} = -2\int \frac{2u\, du}{u^2+1} = -2\ln(u^2+1)$$

$= -2\ln(x^2+2x+2)$ Orig. Int $= 2x - 2\ln(x^2 + 2x + 2) + C$

3. Compute $\int_0^2 (\ln x + x)\,dx$

$$\int_0^2 x\,dx = \frac{x^2}{2}\Big|_0^2 = 2$$

Choose t so $0 < t < 2$

$$\int_t^2 \ln x\,dx = x\ln x\Big|_t^2 - \int_t^2 x\cdot\frac{dx}{x}$$

$u = \ln x \quad dv = dx$

$du = \frac{dx}{x} \quad v = x$

$$= 2\ln 2 - t\ln t - 2 + t$$

$$\rightarrow 2\ln 2 - 2 + 0$$

L'Hop

$$\lim \frac{\ln t}{\frac{1}{t}} = \lim \frac{\frac{1}{t}}{\frac{-1}{t^2}} = \lim -t = 0$$

4. Find length of $y = x^{3/2}, \quad 0 \le x \le 1$

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2} \to \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}$$

$$= \sqrt{1+\frac{9x}{4}} \qquad L = \int_0^1 \sqrt{1+\frac{9x}{4}}\,dx$$

$$= \frac{2}{3}\cdot\frac{4}{9}\left(\left(1+\frac{9x}{4}\right)^{3/2} - 1\right)^{3/2}\Bigg|_0^1$$

$$= \frac{8}{27}\left(\frac{13^{3/2}}{8} - \frac{8}{8}\right)$$

$$= \frac{1}{27}\left(13^{3/2} - 8\right)$$

9. Find $\lim_{n \to \infty} \left(\sqrt{\frac{n^2 + 3n}{4}} - \frac{3n+1}{e^n} \right)$

$$\lim_{n \to \infty} \frac{n^2 + 3n}{4n^2 + 1} = \lim_{x \to \infty} \frac{x^2 + 3x}{4x^2 + 1}$$

$$\underset{\mathcal{L}}{=} \lim_{x \to \infty} \frac{2x + 3}{8x} \underset{\mathcal{L}}{=} \lim_{n \to \infty} \frac{2}{8} = \frac{1}{4}$$

Also $\lim_{x \to \infty} \frac{3x + 1}{e^x} \underset{\mathcal{L}}{=} \lim_{x \to \infty} \frac{3}{e^x} = \frac{3}{\infty} = 0$

$\therefore$ Original limit $= \sqrt{\frac{1}{4}} - 0 = \frac{1}{2}$