7.3 Trig. Substitutions, cont'd.

Ex. Find $\int \frac{dx}{\sqrt{4x^2+9}}$

$\tan^2\theta + 1 = \sec^2\theta$

We want $4x^2 = 9\tan^2\theta$

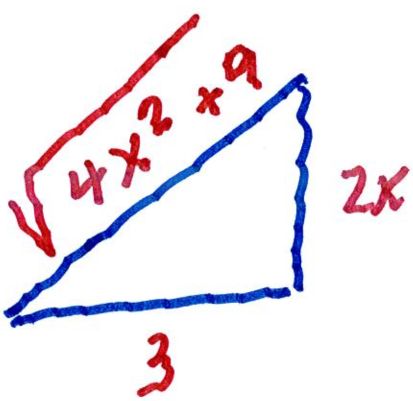
$\rightarrow 2x = 3\tan\theta$, $dx = \frac{3\sec^2\theta\, d\theta}{2}$

$$\int = \int \frac{\frac{3\sec^2\theta\, d\theta}{2}}{\sqrt{9\tan^2\theta + 9}}$$

$$= \frac{3}{2}\cdot\frac{1}{3}\int \frac{\sec^2\theta \, d\theta}{\sec\theta}$$

$$= \frac{1}{2}\int \sec\theta \, d\theta$$

$$= \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$$



$$\frac{2x}{3} = \tan\theta$$

$$\rightarrow \sec\theta = \frac{\sqrt{4x^2+9}}{3}$$

$$\int = \frac{1}{2}\ln\left|\frac{\sqrt{4x^2+9}}{3} + \frac{2x}{3}\right| + C$$

Ex. Find $\int \frac{\sqrt{x^2-4}}{x}\,dx$

Use $\sec^2\theta - 1 = \tan^2\theta$

We want $x^2 = 4\sec^2\theta$

$\rightarrow \underline{x = 2\sec\theta}$, $\underline{dx = 2\sec\theta\tan\theta\,d\theta}$

$$= \int \frac{\sqrt{4\sec^2\theta - 4} \cdot 2\sec\theta\tan\theta\,d\theta}{2\sec\theta}$$

$$= 2\int \tan^2\theta\,d\theta$$

$$= 2\int (\sec^2\theta - 1)\,d\theta$$

$1 + \tan^2\theta = \sec^2\theta$

$\rightarrow \tan^2\theta = \sec^2\theta - 1$

$$\rightarrow = 2\tan\theta - 2\theta + C$$

$\sec\theta = \dfrac{x}{2}$

$$= \sqrt{x^2-4} - 2\sec^{-1}\left(\frac{x}{2}\right) + C$$

$\therefore \tan\theta = \dfrac{\sqrt{x^2-4}}{2}$

$\theta = \sec^{-1}\left(\frac{x}{2}\right)$

$$\sqrt{x^2-4} - 2\sec^{-1}\left(\frac{x}{2}\right) + C$$

This is easy:

$$\int_0^1 \frac{dx}{x^2+1}$$

Use $\tan^2\theta + 1 = \sec^2\theta$

We want $x^2 = \tan^2\theta$

$\rightarrow$ $x = \tan\theta$

$\rightarrow$ $dx = \sec^2\theta \, d\theta$

$0 = \tan\theta$

$\rightarrow \theta = 0$

$1 = \tan\theta$

$\rightarrow \frac{\pi}{4} = \theta$

$$\int = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta \, d\theta}{\tan^2\theta + 1}$$

$$= \int_0^{\frac{\pi}{4}} d\theta = \theta \Big|_0^{\frac{\pi}{4}} = \underline{\underline{\frac{\pi}{4}}}$$

Similarly:

Ex. Find $\displaystyle\int_{1/3}^{\sqrt{3}/3} \frac{dx}{(9x^2+1)^2}$

Use $\tan^2\theta + 1 = \sec^2\theta$

we want $9x^2 = \tan^2\theta$

$\rightarrow 3x = \tan\theta$, $dx = \dfrac{\sec^2\theta \, d\theta}{3}$

Limits: $3 \cdot \frac{1}{3} = \tan\theta \rightarrow \theta = \frac{\pi}{4}$

$3 \frac{\sqrt{3}}{3} = \tan\theta \rightarrow \theta = \frac{\pi}{3}$

$$\int = \int_{\pi/4}^{\pi/3} \frac{\frac{\sec^2\theta \, d\theta}{3}}{\left(\tan^2\theta + 1\right)^2}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{3} \frac{d\theta}{\sec^2\theta} = \frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2\theta \, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\pi/3} \frac{1}{6}\left(1 + \cos 2\theta\right) d\theta$$

$$= \frac{1}{6}\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{1}{6}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) - \frac{1}{6}\left(\frac{\pi}{4} + \frac{1}{2}\right)$$

$$= \frac{\pi}{72} + \frac{\sqrt{3}}{24} - \frac{1}{12}$$

Ex. Find $\displaystyle\int_1^{\sqrt{3}} \frac{x^2\,dx}{\sqrt{4-x^2}}$

Use $1-\sin^2\theta = \cos^2$ Want $x^2 = 4\sin^2\theta$

$\rightarrow x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$

$x=1 \rightarrow 1 = 2\sin\theta$

$\rightarrow \theta = \frac{\pi}{6}$

$x = \sqrt{3}$

$\rightarrow \sqrt{3} = 2\sin\theta$

$\frac{\sqrt{3}}{2} = \sin\theta$

$\theta = \frac{\pi}{3}$

$$\int = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4\sin^2\theta \cdot 2\cos\theta\, d\theta}{\sqrt{4-4\sin^2\theta}}$$

$$= 4 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2\theta \, d\theta$$

$$= 2 \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(1 - \cos 2\theta\right\} d\theta$$

$$= 2\theta - \sin 2\theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{\pi}{3}$$

Ex. Find $\int \frac{x\,dx}{\sqrt{5+2x+x^2}}$

Complete the Square:

$$x^2+2x+5 = (x+1)^2+4$$

$$\int = \int \frac{x\,dx}{\sqrt{4+(x+1)^2}}$$

Set $u = x+1$

$du = dx$

$u-1 = x$

$$= \int \frac{(u-1)\,du}{\sqrt{4+u^2}}$$

$$= \int \frac{u\,du}{\sqrt{4+u^2}} - \int \frac{du}{\sqrt{4+u^2}}$$

$$v = 4+u^2$$

$$dv = 2u\,du$$

$$\frac{dv}{2} = u\,du$$

$$\int \frac{\frac{1}{2}\,dv}{\sqrt{v}}$$

$$= 2 \cdot \tfrac{1}{2}\sqrt{v}$$

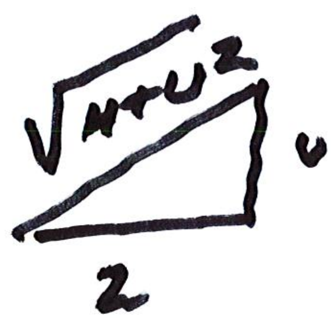
$$= \sqrt{4+u^2} = \sqrt{4+(x+1)^2}$$

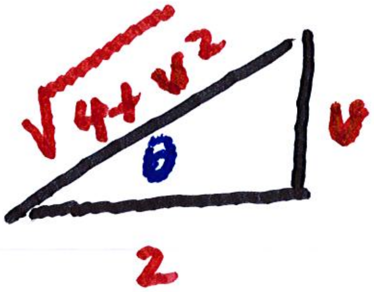
$$u = 2\tan\theta$$

$$du = 2\sec^2\theta\,d\theta$$

$$= -\int \frac{2\sec^2\theta\,d\theta}{2\sec\theta}$$

$$= -\ln|\sec\theta + \tan\theta|$$





$$\tan\theta = \frac{v}{2}$$

$$\rightarrow \sec\theta = \frac{\sqrt{4+v^2}}{2}$$

$$\int = -\ln\left|\frac{\sqrt{4+v^2}}{2} + \frac{v}{2}\right|$$

$$= -\ln\left|\frac{\sqrt{4+(x+1)^2}}{2} + \frac{(x+1)}{2}\right|$$

$\therefore$ Original Integral

$$= \sqrt{4+(x+1)^2} - \ln\left|\frac{\sqrt{4+(x+1)^2}}{2} + \frac{x+1}{2}\right| + C$$

Ex. A bit tricky:

$$\int_{2+\sqrt{2}}^{2+\sqrt{3}} \frac{dx}{\sqrt{4x-x^2}}$$

What's the constant?

Comp the square:

$$x^2-4x = (x-2)^2-4$$

$$\therefore\ 4x-x^2 = 4-(x-2)^2$$

$$\int_{2+\sqrt{2}}^{2+\sqrt{3}} \frac{dx}{\sqrt{4-(x-2)^2}}$$

Set $u = x-2$

$\rightarrow du = dx$

$x = 2+\sqrt{3}$

$\rightarrow u = (2+\sqrt{3}) - 2 \rightarrow u = \sqrt{3}$

$x = 2+\sqrt{2}$

$\rightarrow u = (2+\sqrt{2}) - 2 \rightarrow u = \sqrt{2}$

$$\therefore \int = \int_{\sqrt{2}}^{\sqrt{3}} \frac{du}{\sqrt{4-u^2}}$$

Now set $\underline{u = 2\sin\theta}$.

Then $\underline{du = 2\cos\theta \, d\theta}$

$u = \sqrt{3}$

$\sqrt{3} = 2\sin\theta$

$\frac{\sqrt{3}}{2} = \sin\theta$

$\theta = \frac{\pi}{3}$

$u = \sqrt{2}$

$\sqrt{2} = 2\sin\theta$

$\frac{\sqrt{2}}{2} = \sin\theta$

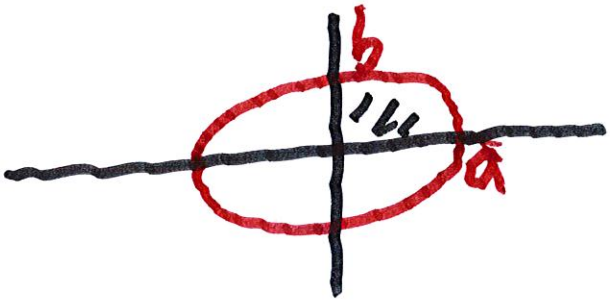
$\theta = \frac{\pi}{4}$

$$\int_{\sqrt{2}}^{\sqrt{3}} \frac{du}{\sqrt{4-u^2}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\cos\theta\, d\theta}{\sqrt{4-4\sin^2\theta}}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\theta = \frac{\pi}{12}$$

Ex. Compute area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\therefore A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

So we compute $\displaystyle\int_0^a \sqrt{a^2 - x^2}\, dx$

$1 - \sin^2\theta = \cos^2\theta$

We want $x^2 = a^2 \sin^2\theta$

$\rightarrow \underline{x = a \sin\theta} \rightarrow dx = a\cos\theta\, d\theta$

When $\theta = 0$, $x = 0$

When $\theta = \frac{\pi}{2}$, $x = a$

$$\int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2 \sin^2\theta} \cdot a\cos\theta \, d\theta$$

$$= a^2 \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

$$= a^2 \int_0^{\frac{\pi}{2}} \frac{(1 + \cos 2\theta)}{2} \, d\theta$$

$$= \frac{a^2}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2} + 0\right) - 0$$

$$= \frac{a^2\pi}{4}$$

$$\therefore \text{Area} = \frac{4b}{a} \cdot \frac{a^2\pi}{4} = \underline{\underline{\pi ab}}$$

Ex. Compute $\int \sqrt{7-2x^2}\, dx$

Id. is $1-\sin^2\theta = \cos^2\theta$

We want $2x^2 = 7\sin^2\theta$

$\rightarrow x^2 = \frac{7}{2}\sin^2\theta$

$x = \sqrt{\frac{7}{2}}\sin\theta \qquad dx = \sqrt{\frac{7}{2}}\cos\theta\, d\theta$

$$\int \sqrt{7-7\sin^2\theta} \cdot \sqrt{\frac{7}{2}}\cos\theta\, d\theta$$

$$= \frac{7}{\sqrt{2}}\int \cos^2\theta\, d\theta = \frac{7}{2\sqrt{2}}\int 1+\cos 2\theta\, d\theta$$