7.6 Using Tables of Integrals

All tables use expressions like

$\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$, $\sqrt{u^2-a^2}$

instead of $\sqrt{4x^2-6x+7}$

$$\int \sqrt{4x^2-6x+7}\,dx$$

$$4x^2-6x+7 = 4\left(x^2-\frac{3}{2}x+\frac{7}{4}\right)$$

$$= 4\left(\left(x-\frac{3}{4}\right)^2+\left(\frac{7}{4}-\frac{9}{16}\right)\right)$$

$$= 4\left((x-\tfrac{3}{4})^2 + \frac{19}{16}\right)$$

Now set $u = \left(x - \frac{3}{4}\right) \rightarrow du = dx$

$$\therefore \int \sqrt{4x^2 - 6x + 7}\, dx$$

$$= \int 2\sqrt{u^2 + \frac{19}{16}}$$

$$= 2\int \sqrt{u^2+a^2}\,du, \quad \text{where } a = \frac{\sqrt{19}}{4}$$

$$\text{or } a^2 = \frac{19}{16}$$

Formula
#21 states:

$$\int \sqrt{u^2+a^2}\,du$$

$$= \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right)+C$$

Replace $a^2 = \frac{19}{16}$

Then set $u = x - \frac{3}{4}$

} gives integral

Integral is

$$= \left(x - \frac{3}{4}\right)\sqrt{\frac{19}{16} + \left(x - \frac{3}{4}\right)^2}$$

$$+ \frac{19}{16} \ln\left(x - \frac{3}{4} + \sqrt{\frac{19}{16} + \left(x - \frac{3}{4}\right)^2}\right)$$

$$+ C$$

Reduction Formulas

$$\int x^4 \cos x = ?$$

Formula 85

$$\int u^n \cos u = u^n \sin u - n \int u^{n-1} \sin u \, du$$

If $n = 4$

$$\int x^4 \cos x \, dx = x^4 \sin x - 4 \int x^3 \sin x \, dx$$

Now use # 84 with $n = 3$:

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3 \int x^2 \cos x \, dx$$

$$\therefore \int x^4 \cos x \, dx = x^4 \sin x$$

Rule 85 $(n=2)$

$$+ 4x^3 \cos x - 12 \int x^2 \cos x \, dx$$

etc. ...

$$= x^4 \sin x + 4x^3 \cos x$$

$$- 12x^2 \sin x + 24 \int x \sin x \, dx$$

Rule 82

$$= x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x$$

$$+ 24(\sin x - x\cos x) + C$$

Ex. Compute $\int_0^4 \sin(\sqrt{x})\,dx$

Set $x = y^2$ $(\sqrt{x} = y)$ $dx = 2y\,dy$

$= \int_0^2 \sin y \; 2y\,dy$ (Int. by Parts

$u = 2y$ $dv = \sin y\,dy$

$du = 2\,dy$ $v = -\cos y$

$= -2y\cos y \Big|_0^2 + \int_0^2 2\cos y\,dy$

$= -4\cos 2 + 2\sin 2 + K$

Clever Manipulations:

$$\int \frac{dx}{e^x + 1} = \int \frac{e^x\,dx}{e^{2x} + e^x}$$

Set $u = e^x$ $\quad du = e^x\,dx$

$$= \int \frac{du}{u^2 + u} = \int \frac{1}{u} - \frac{1}{u+1}\,du$$

$$= \ln|u| - \ln|u+1| + K$$

$$= \ln e^x - \ln(e^x + 1) + K$$

$$= \ln\left(\frac{e^x}{e^x + 1}\right) + K$$

7.7 Approximate Integration

For some integrals, we have to estimate the integral.

Ex $\displaystyle\int_0^1 e^{x^2}\,dx$ and $\displaystyle\int_{-1}^{2} \sqrt{1+x^3}\,dx$

cannot be computed exactly.

In general, consider

$$\int_a^b f(x)\,dx$$

We partition $[a, b]$:

$a = x_0 < x_1 \dots < x_n = b,$

where $x_i - x_{i-1} = \Delta x = \dfrac{b-a}{n}$

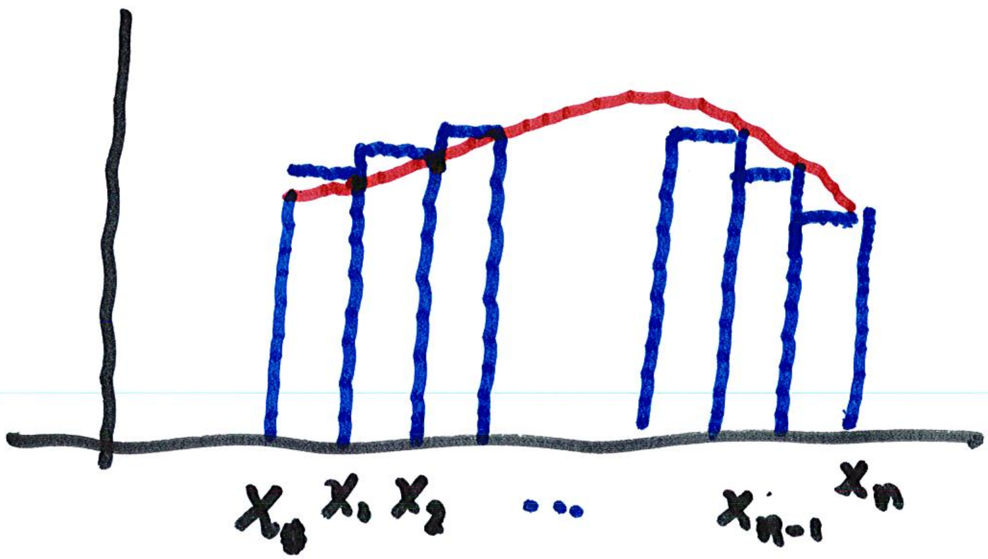
for all $i = 1, 2, \dots, n$.

We can approx $\displaystyle\int_a^b f(x)\,dx$ by

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x \leftarrow \text{called}$$

the right

endpoint approximation



Right Endpoint Approximation

Height = $f(x_i)$ for $[x_{i-1}, x_i]$

Left Endpoint Approx

Height = $f(x_{i-1})$ for $[x_{i-1}, x_i]$

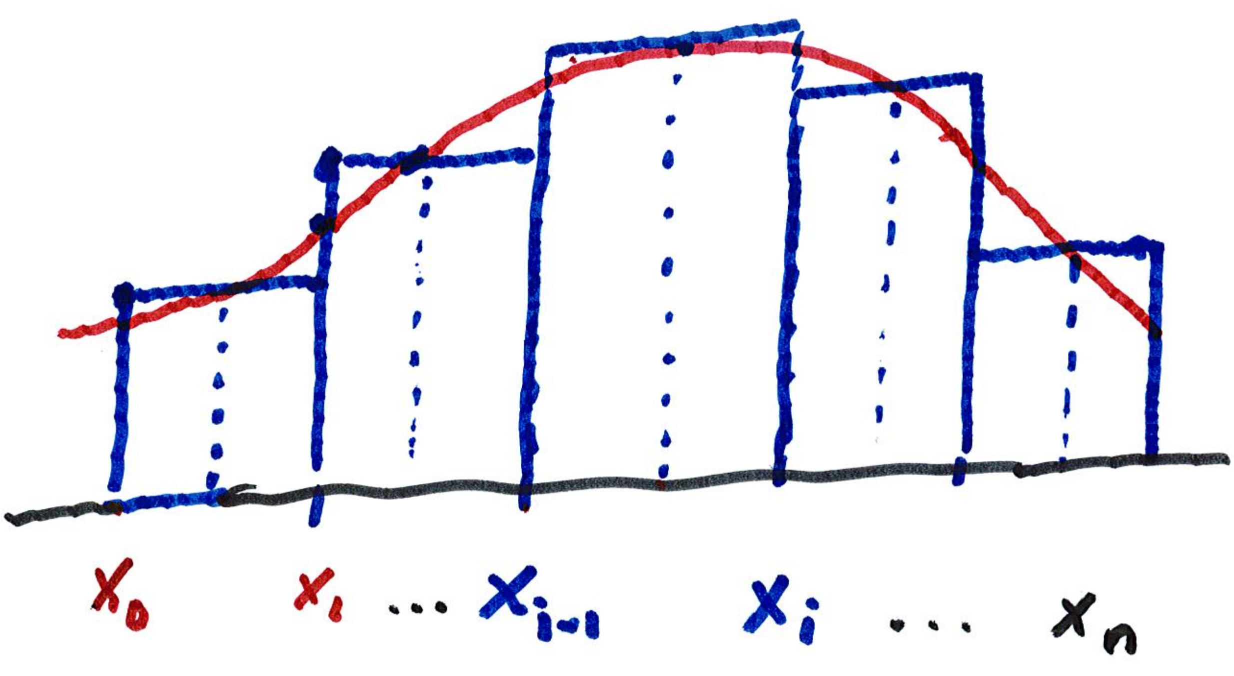
We get

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

The Midpoint Rule is usually better:

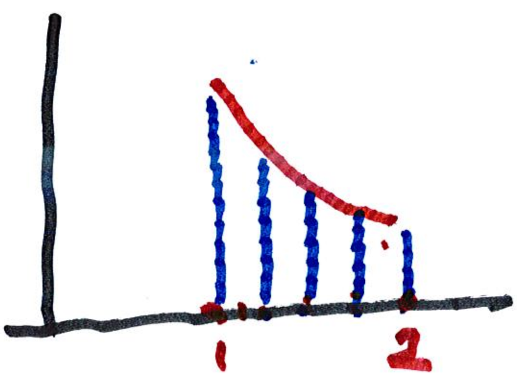
$$M_n = \left[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n) \right] \Delta x$$

where $\bar{x}_i = \dfrac{x_{i-1} + x_i}{2}$.



Usually one part of rectangle is too high and one is too low.

Ex. Use Mid.-Pt. Rule to approx. $\ln 2$ with $n = 4$



$$\ln 2 = \int_1^2 \frac{dx}{x}$$

$$\begin{array}{ccccccccc} x_0 & < & x_1 & < & x_2 & < & x_3 & = & x_4 \\ \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\ 1 & & 5/4 & & \frac{3}{2} & & \frac{7}{4} & & 2 \end{array}$$

Midpoints are

$$\frac{9}{8}, \quad \frac{11}{8}, \quad \frac{13}{8}, \quad \frac{15}{8}$$

$$M_4 = \left[\frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15} \right] \cdot \frac{1}{4}$$

$$f(x) = \frac{1}{x}$$

$$M_4 \approx \underline{\underline{.69122}}$$

With a calculator

$$\ln 2 \approx \underline{\underline{.69315}}$$

$$\ln 2 - M_4 \approx .0023$$

Another rule is the

Trapezoidal Rule.

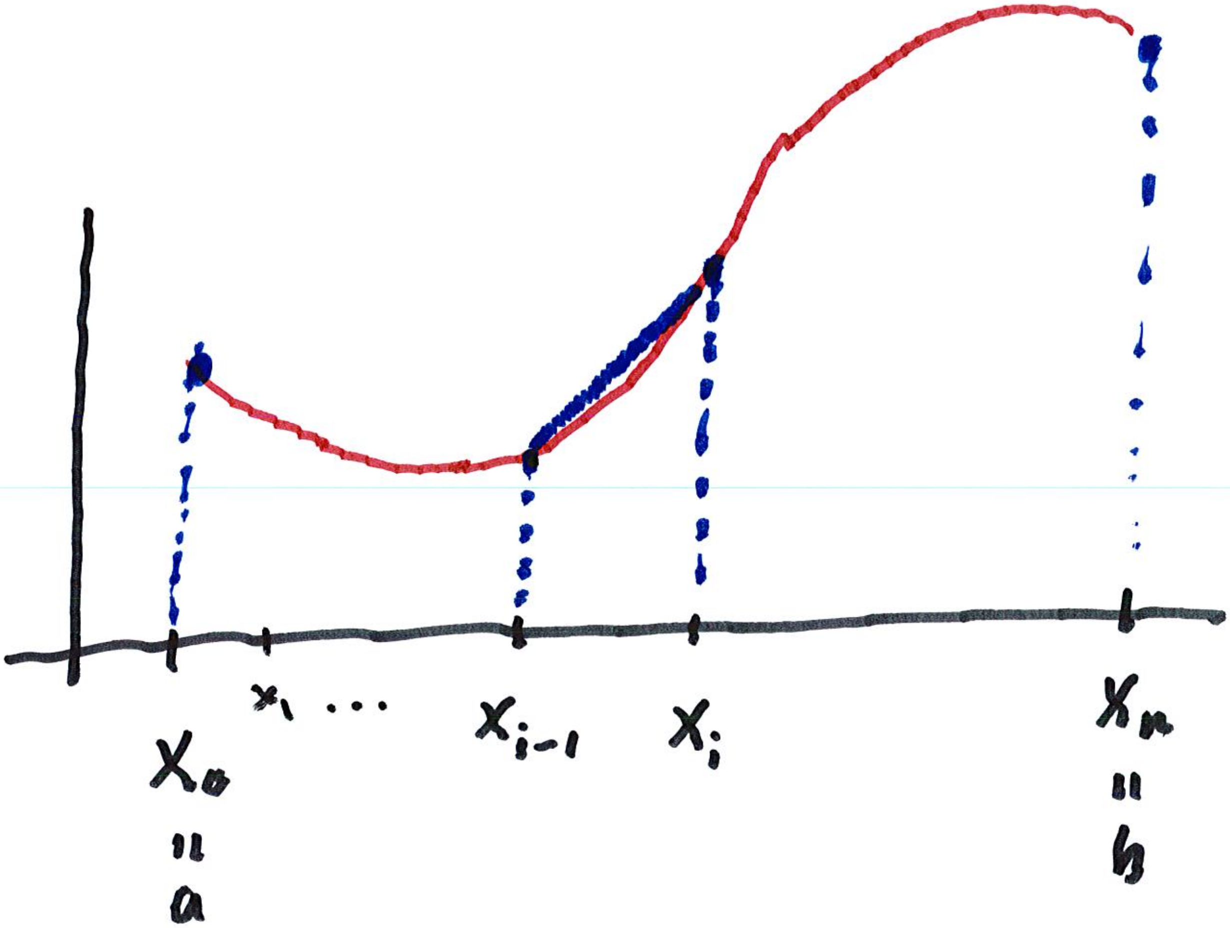
We take the average of

the Right End. and

Left End. rules

$$\Delta x\left(\frac{f(x_0)+f(x_1)}{2}+\frac{f(x_1)+f(x_2)}{2}+\frac{f(x_2)+f(x_3)}{2}+\dots+\frac{f(x_{n-1})+f(x_n)}{2}\right)$$

7.1



$$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n) \right] = T_n$$

where $\Delta x = \dfrac{b-a}{n}$ and

$x_i = a + i\Delta x$.

Use Trap. Rule with $n = 4$

to approximate $\displaystyle\int_1^2 \frac{1}{x}\,dx = \ln 2$

$$T_4 = \frac{1}{8}\left[\frac{1}{1} + \frac{2\cdot 4}{5} + \frac{2\cdot 4}{6} + \frac{2\cdot 4}{7} + \frac{1}{2}\right]$$

Using a calculator:

$$T_4 \approx .69702$$

$$T_4 - \ln 2 \approx .00387$$

We have estimates for both E_M and E_T:

Suppose that

$$|f''(x)| \le K \text{ for all } x \text{ with } a \le x \le b.$$

Then set

$$E_T = \int_a^b f(x)\,dx - T_n$$

and $$E_M = \int_a^b f(x)\,dx - M_n$$

$$|E_T| \leq \frac{K_2(b-a)}{12n^2}$$

and

$$|E_M| \leq \frac{K_2(b-a)^3}{24n^2}$$

where K_2 = Max Value of $|f''(x)|$

How big is $|E_T|$ if

n is large? (for $f(x) = \frac{1}{x}$)

$$f(x) = \frac{1}{x} \to f'(x) = -\frac{1}{x^2}$$

$$\to f''(x) = \frac{2}{x^3}.$$

$$\therefore |f''(x)| \le 2 \quad \text{if } 1 \le x \le 2.$$

i.e., $K_2 = 2$

$$\therefore E_M \le \frac{2 \cdot 1^3}{12 n^2}$$

for $\int_1^2 \frac{dx}{x}\, dx$

$$\text{or } E_T \le \frac{1}{6n^2}$$

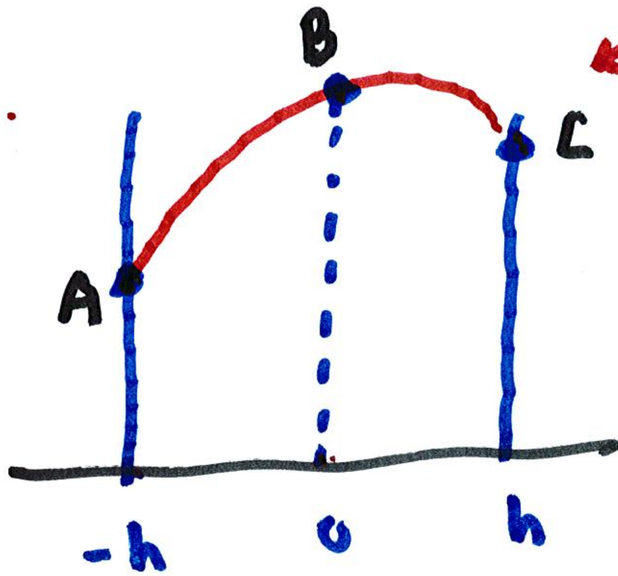
$\therefore$ If we set $n = 100$,

$$|E_T| \leq \frac{1}{6 \cdot n^2} = \frac{1}{60000}$$

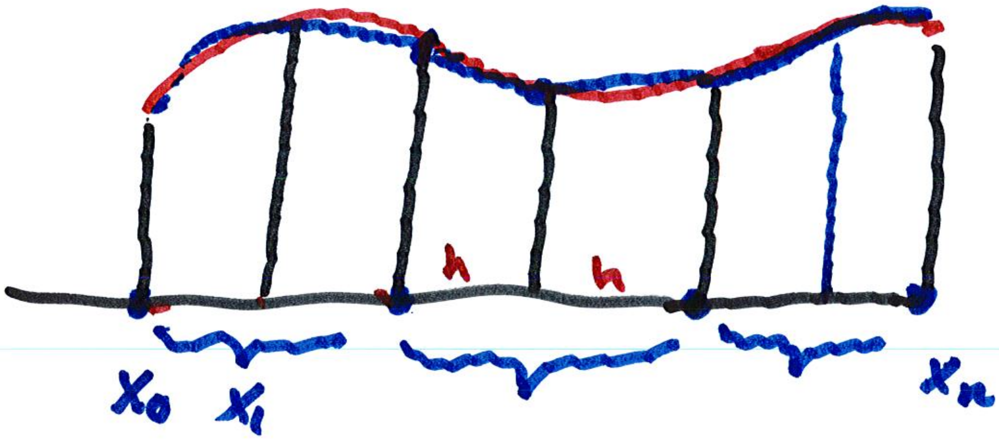
Similarly,

$$|E_M| \leq \frac{1}{12n^2} = \frac{1}{120{,}000}$$

Simpson's Rule



A parabola through A, B, and C



$$\frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2) \right]$$

$$+ \frac{h}{3}\left[f(x_2) + 4f(x_3) + f(x_4) \right]$$

$$+$$

$$\vdots \quad \frac{h}{3}\left[f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n}) \right]$$

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n) \right]$$

where n must be even.

For $\ln 2 = \int_1^2 \frac{1}{x}\,dx$

$$S_4 = \frac{1}{3\cdot 4}\left[\frac{1}{1} + \frac{4\cdot 4}{5} + \frac{2\cdot 4}{6} + \frac{4\cdot 4}{7} + \frac{1}{2}\right]$$

$$= .693254\ldots$$

Recall $\ln 2 \approx .69315$,

so $E_S \approx .00010$ with $n=4$

Error Estimate for Simpson's Rule.

Suppose that $|f^{(4)}(x)| \leq K_4$

For $\ln 2 = \int_1^2 \frac{dx}{x}$,

$b - a = 2 - 1 = 1$, and

if $f(x) = \frac{1}{x}$,

$$f'(x) = -\frac{1}{x^2}$$

$$\vdots$$

$$f^{(4)}(x) = \frac{24}{x^5}, \text{ so}$$

$K_4 \leq 24$

If we set $n = 100$, then

$$|E_S| \leq \frac{24}{180\, n^4} < \frac{1}{7 \times 10^8}$$

$$\therefore \left| S_{100} - \ln 2 \right| < \frac{1}{7 \times 10^8}$$

The general estimate for Simpson's Rule is

$$|E_S| \leq \frac{K_4 (b-a)}{180\, n^4}$$