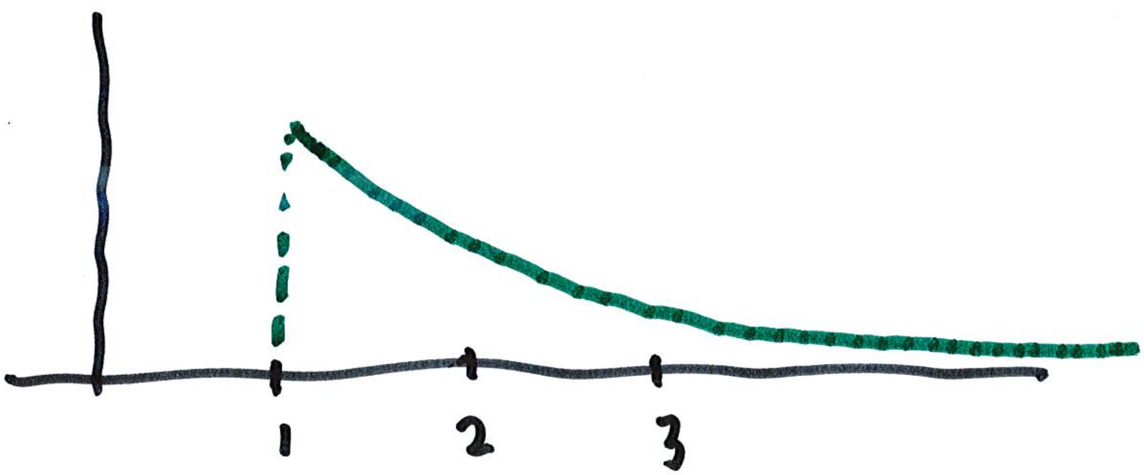
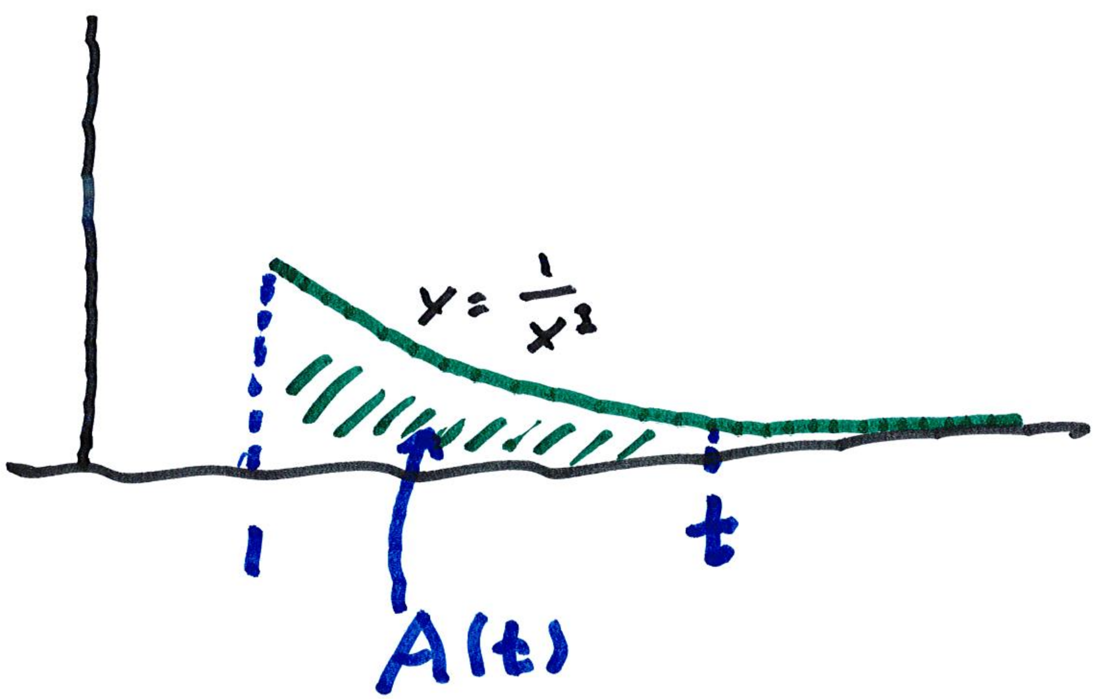
7.8 Improper Integrals

Ex. What is the area under the curve $y = \frac{1}{x^2}$ for $1 \leq x < \infty$



Idea: Choose any t, for $t > 1$, and let $A(t)$ = area under $y = \frac{1}{x^2}$ for $1 \le x \le t$.

Then let $t \to \infty$.

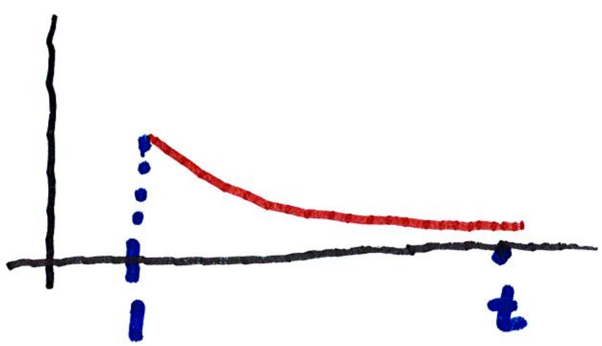


$$A(t) = \int_1^t \frac{dx}{x^2}$$

$$= -\frac{1}{x}\Big|_1^t = -\frac{1}{t} - \left(-\frac{1}{1}\right)$$

$$= 1 - \frac{1}{t}$$

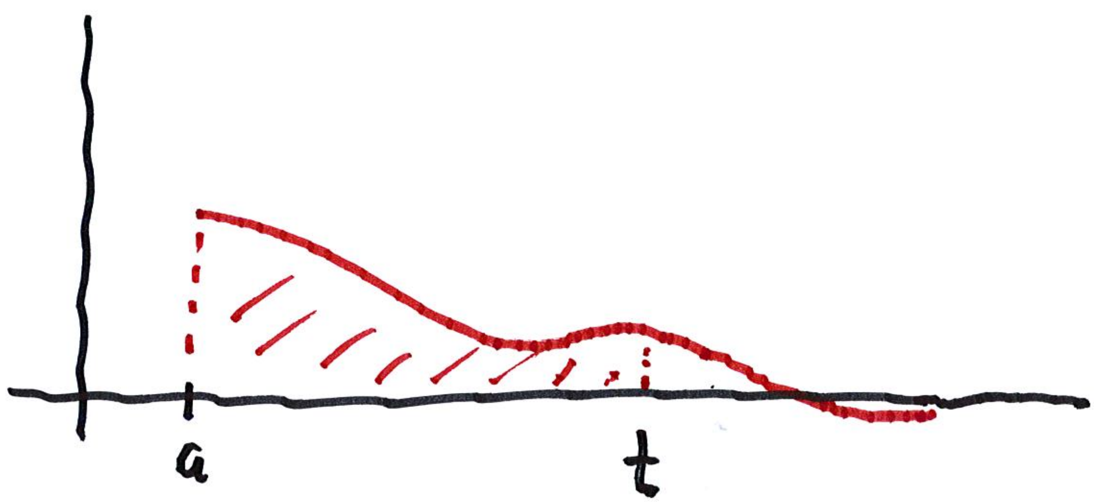
$$\lim_{t \to \infty} 1 - \frac{1}{t} = 1 - 0 = 1$$



Area under

$y = \frac{1}{x^2}$ is $= 1$

Def'n : 1. Suppose $\displaystyle\int_a^t f(x)\,dx$ exists for all $t \geq a$.



Suppose $\displaystyle\lim_{t\to\infty}\int_a^t f(x)\,dx$ exists and is finite.

Then we set

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx .$$

We say

$$\int_a^{\infty} f(x)\,dx \quad \text{and} \quad \int_{-\infty}^{b} f(x)\,dx$$

are **convergent** if the $\int_t^b f(x)\,dx$ corresponding limit exists

and **divergent** if the limit does not exist.

Ex. Is $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x}}$ convergent

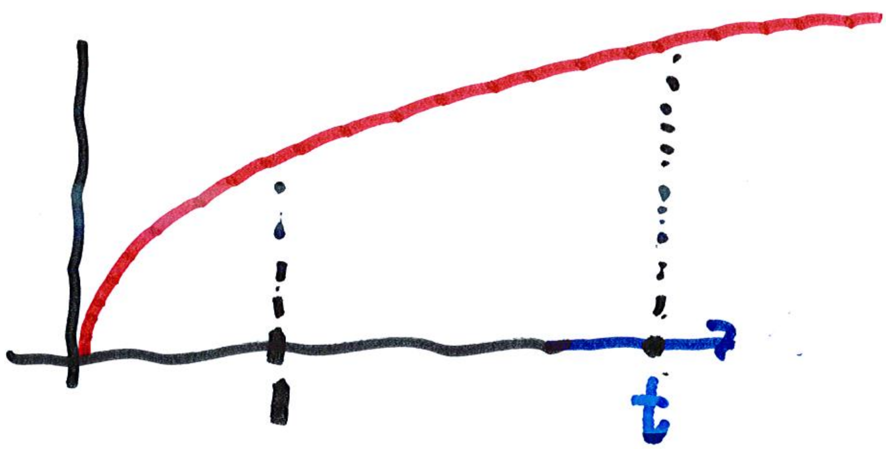
or divergent?

Let $t > 1$

$$\int_1^t \frac{dx}{\sqrt{x}} = 2\sqrt{x}\,\Big|_1^t$$

$$= 2\sqrt{t} - 2$$

$\lim \sqrt{t} = \infty$ as $t \to \infty$



$\therefore$ $2\sqrt{t} \to \infty$, as does $2\sqrt{t} - 2$

$\therefore$ $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$ is divergent

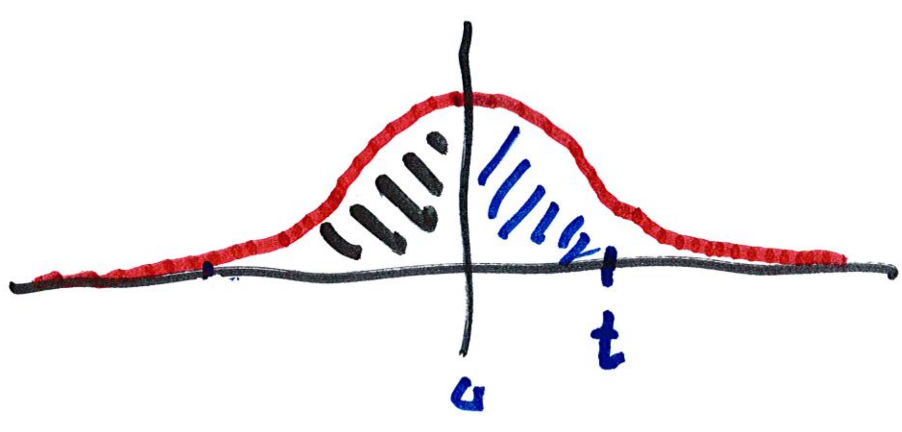
We can also say,

the area under $y = \frac{1}{\sqrt{x}}$,

is infinite for $1 \le x \le \infty$

Ex. What is the area under

$y = \frac{1}{1+x^2}$ for $-\infty < x < \infty$

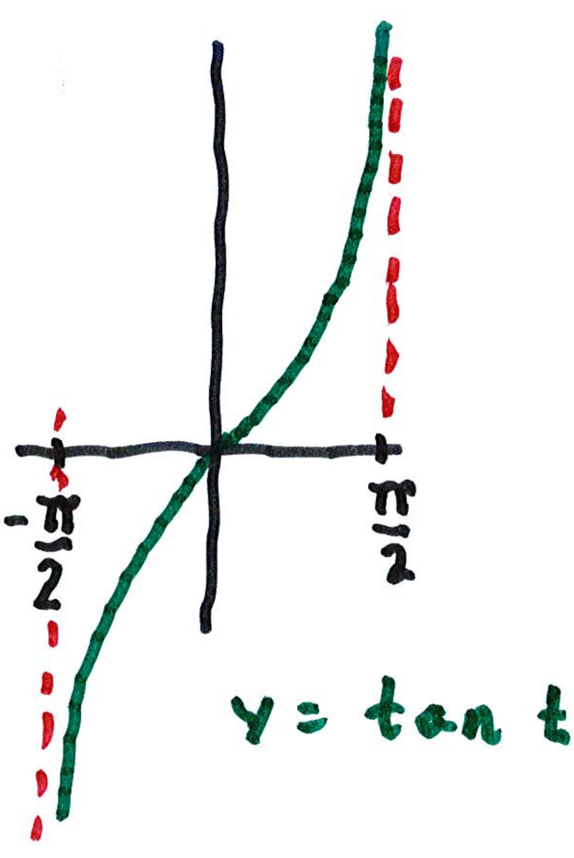


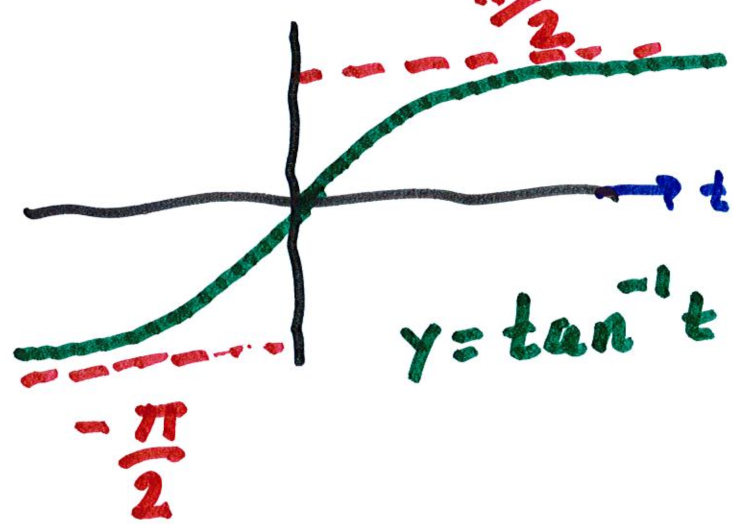
Look at $\int_0^\infty \frac{dx}{1+x^2}$.

If $t \geq 0$,

$$\int_0^t \frac{1}{1+x^2}\,dx = \tan^{-1}x \Big|_0^t$$

$$= \tan^{-1}t - 0 = \tan^{-1}t$$





$$\therefore \lim_{t \to +\infty} \tan^{-1} t = \frac{\pi}{2}$$

$$\left(\text{and } \lim_{t \to -\infty} \tan^{-1} t = -\frac{\pi}{2} \right)$$

$$\therefore \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$$

Also, for $t \le x \le 0$,

$$\int_t^0 \frac{dx}{1+x^2} = \tan^{-1} x \Big|_t^0 = -\tan^{-1} t$$

and $\lim_{t \to -\infty} -\tan^{-1} t$

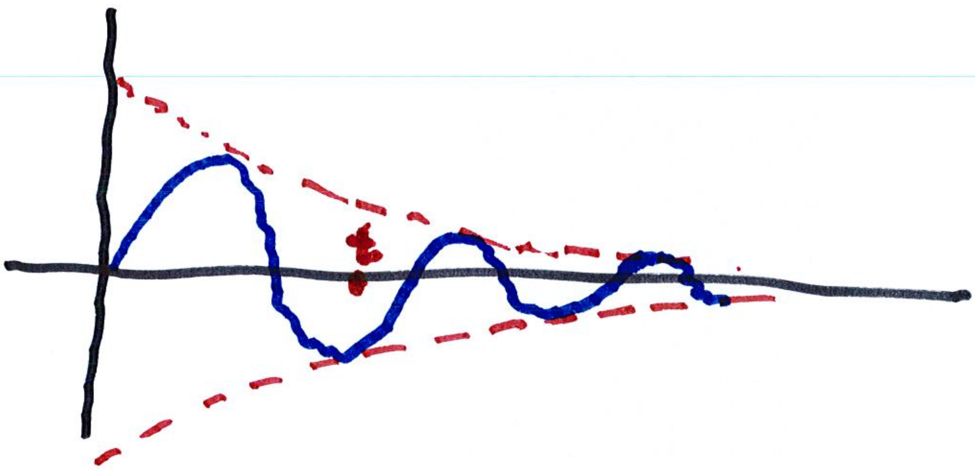
$$= -\lim_{t \to -\infty} \tan^{-1} t = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Since $\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$

and $\int_{-\infty}^{0} \frac{dx}{1+x^2} = \frac{\pi}{2}$, we conclude

that $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

Ex. $\int_0^\infty e^{-x} \sin x \, dx$ convergent ?



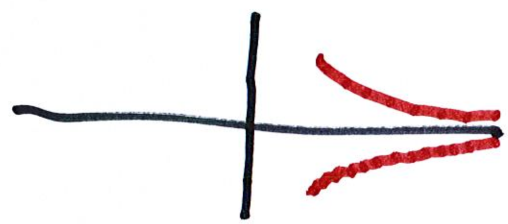
For $t \geq 0$,

$$\int_0^t e^{-x} \sin x \, dx = \frac{e^{-x}}{2}\left(-\sin x - \cos x\right)\Bigg|_0^t$$

(by Formula 98 in Table)

$$= \frac{-e^{-t}}{2}(\sin t + \cos t) + \frac{1}{2}$$

Note that



$$-e^{-t} \leq e^{-t}\sin t \leq e^{-t}$$

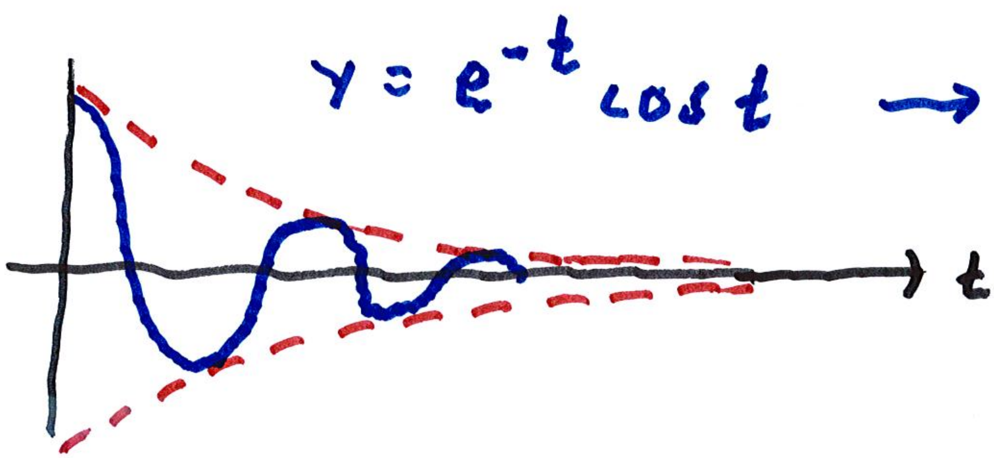
and

$$-e^{-t} \leq e^{-t}\cos t \leq e^{-t}$$

$\therefore$ The Squeeze Thm $\Rightarrow$

$$\lim_{t \to \infty} e^{-t} \sin t = 0 \qquad \text{and}$$

$$\lim_{t \to \infty} e^{-t} \cos t = 0.$$



$$\therefore \lim_{t \to \infty} \left(-\frac{e^{-t}}{2} (\sin t + \cos t) + \frac{1}{2} \right)$$

$$= \frac{1}{2}$$

$$\therefore \int_0^\infty e^{-t} \sin t \, dt = \frac{1}{2}$$

This question is VERY IMPORTANT.

For what p, $0 < p < \infty$,

is $\displaystyle\int_1^\infty \frac{dx}{x^p}$ convergent?

Suppose $p \neq 1$.

P # 1

$$\int_1^t \frac{dx}{x^p} = \int_1^t x^{-p}\,dx$$

$$= \frac{1}{1-p}\, x^{1-p} \Big|_1^t = \frac{1}{1-p}\left(t^{1-p} - 1 \right)$$

$$\lim_{t \to \infty} t^{1-p} = \infty \quad (\text{if } 0 < p < 1)$$

and

$$\lim_{t \to \infty} t^{1-p} = 0 \quad \text{if} \quad 1 < p < \infty$$

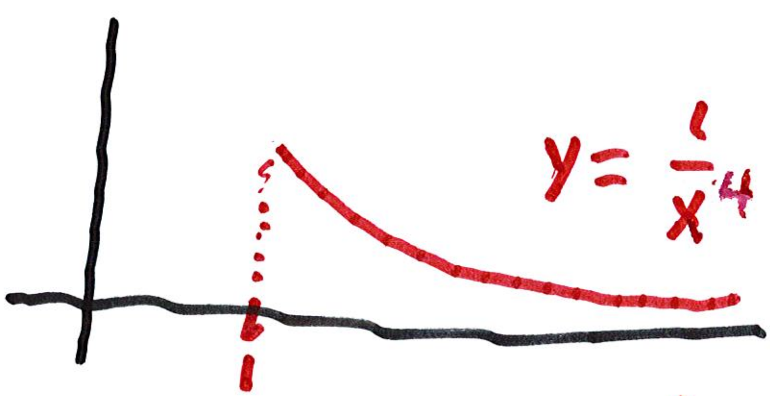
Since $1-p < 0$

$\therefore$ If $0 < p < 1$, then

$$\int_1^\infty \frac{dx}{x^p} \text{ diverges}$$

and if $1 < p < \infty$, then

$$\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1} \quad \text{converges}$$



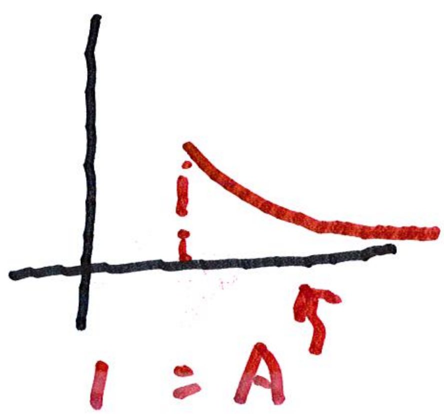
$p = 4$

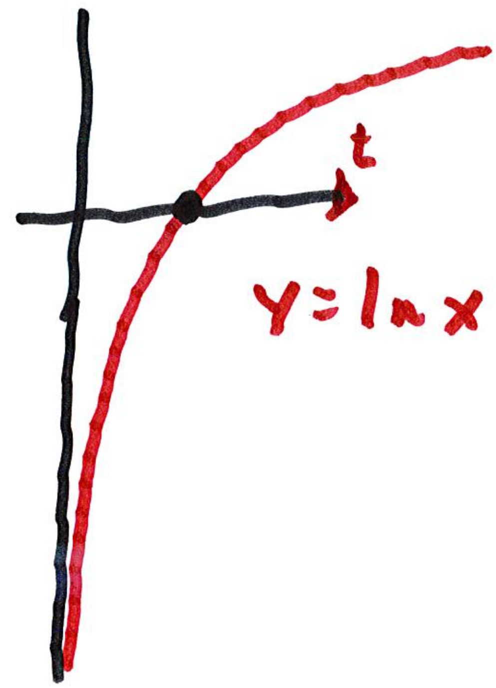
$A = \frac{1}{3}$

If $p = 1$

$$\int_1^t \frac{dx}{x} = \ln x \Big|_1^t = \ln t \longrightarrow \infty$$

as $t \to \infty$





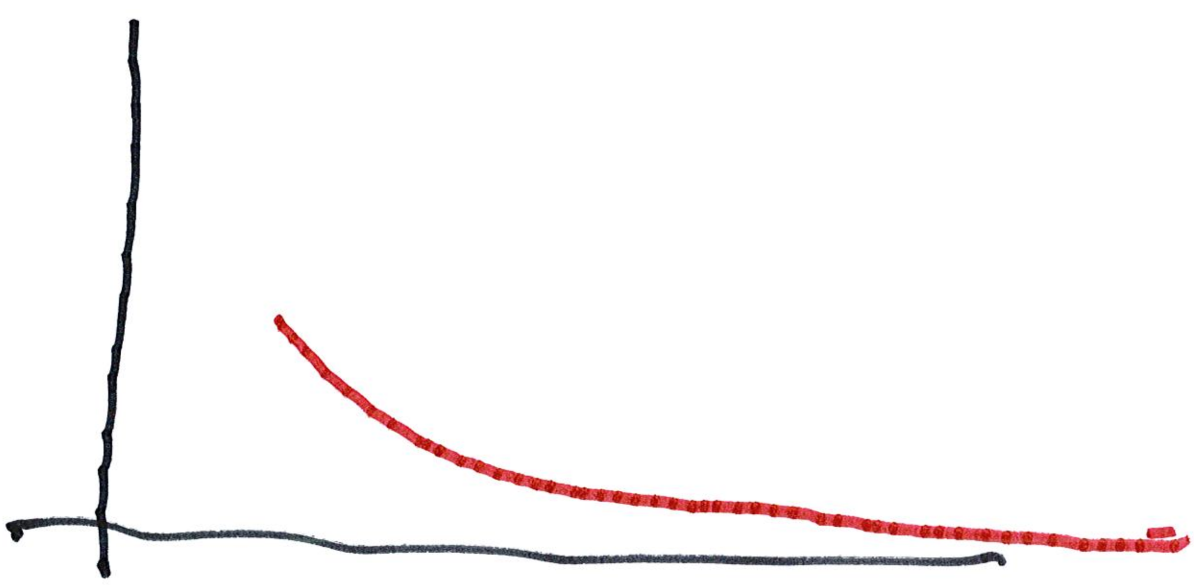
$\therefore \int_1^\infty \frac{dx}{x}$ also diverges if $p = 1$

Ex. $\displaystyle\int_1^\infty \frac{dx}{x^{3/2}}$ is convergent,

$\left(\frac{3}{2} > 1\right)$

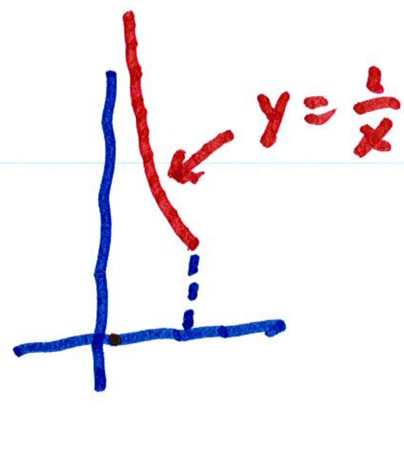
$\displaystyle\int_1^\infty \frac{dx}{x^{2/3}}$ is divergent

$\left(\frac{2}{3} < 1\right)$

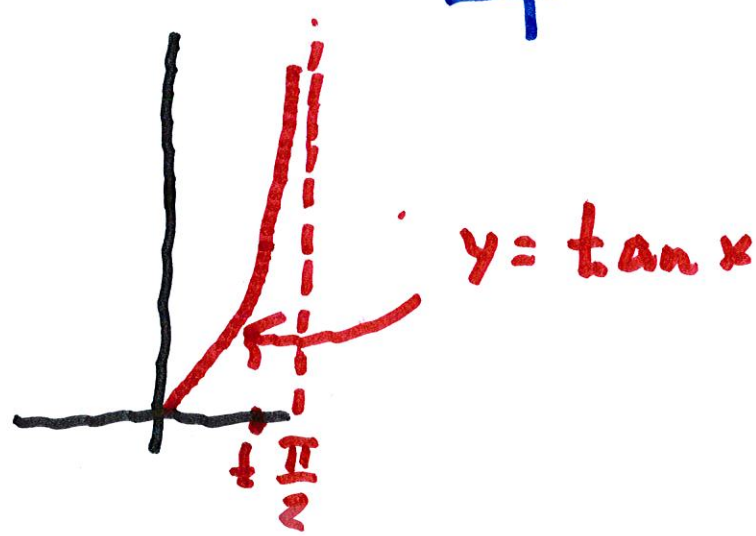


Another improper integral

would be $\int_0^1 \frac{dx}{x}$



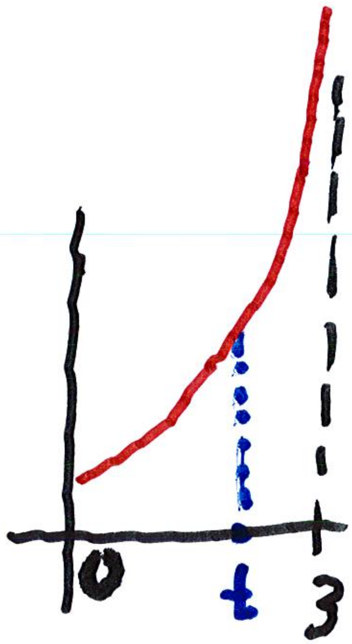
or $\int_0^{\pi/2} \tan x$



If f is discontinuous at b,

then we set $\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$

Ex. Does $\displaystyle\int_0^3 \frac{dx}{\sqrt{3-x}}$ converge?



$$\int_0^t \frac{dx}{\sqrt{3-x}} = -2\sqrt{3-x}\,\Big|_0^t$$

$$= -2\sqrt{3-t} - \left(-2\sqrt{3}\right)$$

$$\to 0 + 2\sqrt{3} = \underline{\underline{2\sqrt{3}}}$$

as $t \to 3^-$ Integral Converges

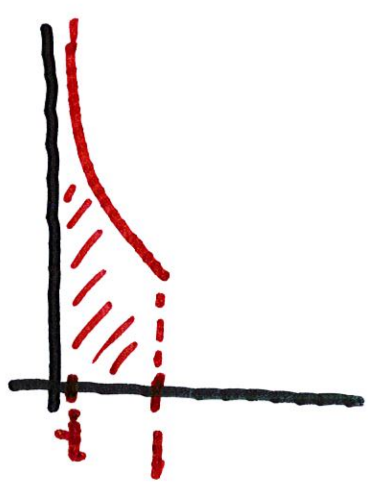
If $0 < p < 1$,

$$\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p} \quad \text{converges}$$

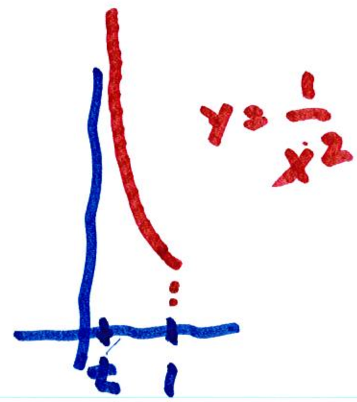
and

$$\int_0^1 \frac{dx}{x^p} \quad \text{diverges if } p > 1$$

Ex. $$\int_0^1 \frac{dx}{x^{2/3}} = \frac{1}{1-\frac{2}{3}} = \frac{1}{\frac{1}{3}} = 3$$



Ex. What about $\int_0^1 \frac{dx}{x^2}$



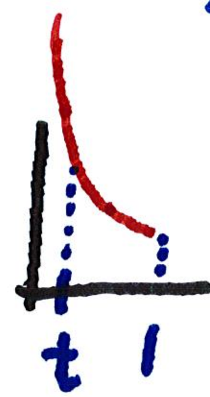
$$\int_t^1 \frac{dx}{x^2} = -\frac{1}{x}\Big|_t^1$$

$$= -\frac{1}{1} - \left(-\frac{1}{t}\right) = \frac{1}{t} - 1$$

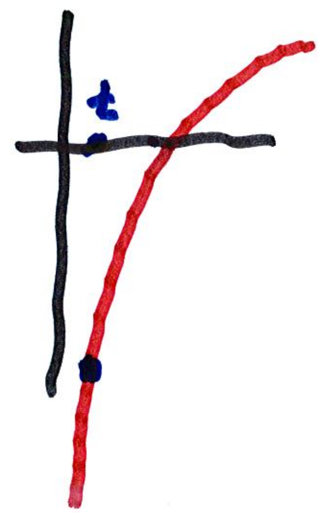
As $t \to 0^+$, $\lim_{t\to\infty} \frac{1}{0^+} = \infty$.

$\therefore \lim_{t\to 0} \frac{1}{t} - 1 = \infty$ INTEGRAL DIVERGES

Ex. $\int_0^1 \frac{dx}{x}$ also diverges:



$$\int_t^1 \frac{dx}{x} = \ln x \Big|_t^1 = 0 - \ln t \to \infty$$



$\therefore$ INT. DIVERGES.

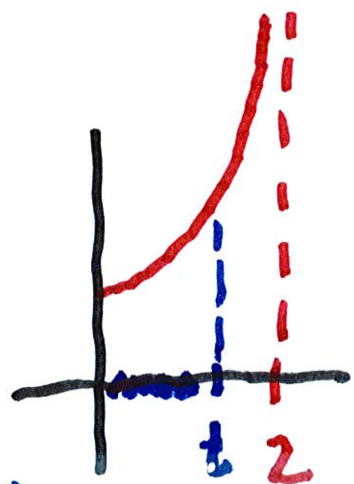
Ex. $$\int_0^2 \frac{dx}{\sqrt{4-x^2}}$$

$x = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$

$$\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{2\cos\theta \, d\theta}{2\cos\theta}$$

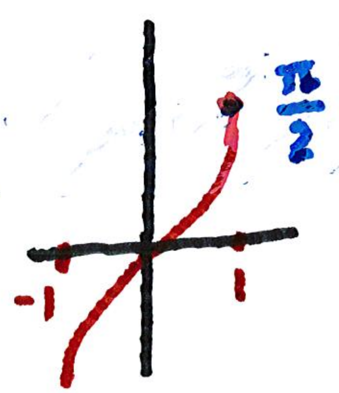
$$= \int d\theta = \theta = \sin^{-1}\left(\frac{x}{2}\right)$$

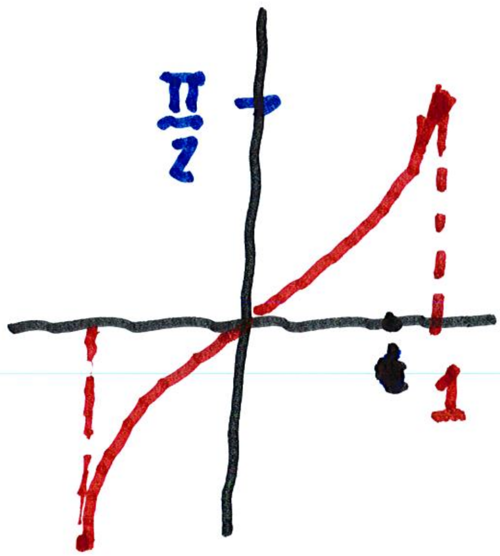


Suppose $t < 2$

$$\int_0^t \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\left(\frac{x}{2}\right)\Big|_0^t \rightarrow \frac{\pi}{2}$$

as $t \rightarrow 2$

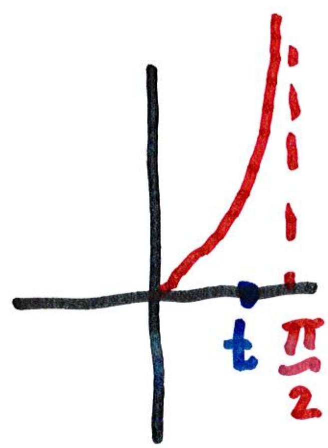




$y = \sin^{-1} x$

$$\lim_{t \to 2} \sin^{-1}\left(\frac{t}{2}\right) = \sin^{-1}(1) = \frac{\pi}{2}$$

Ex. $\displaystyle\int_0^{\frac{\pi}{2}} \tan\theta \, d\theta$

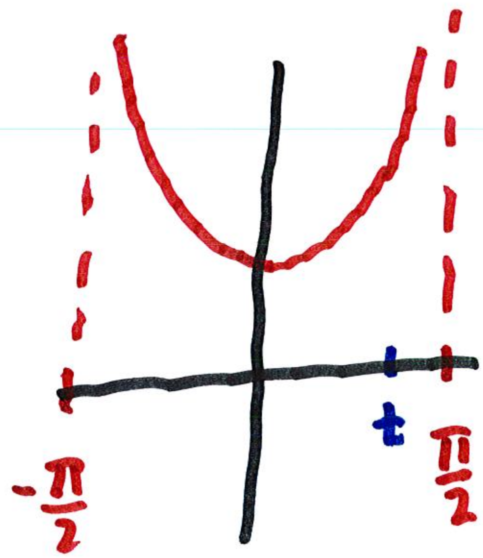


$$\int_0^t \tan\theta \, d\theta = \ln|\sec\theta| \Big]_0^t$$

$$\to \infty \text{ as } t \to \frac{\pi}{2}$$

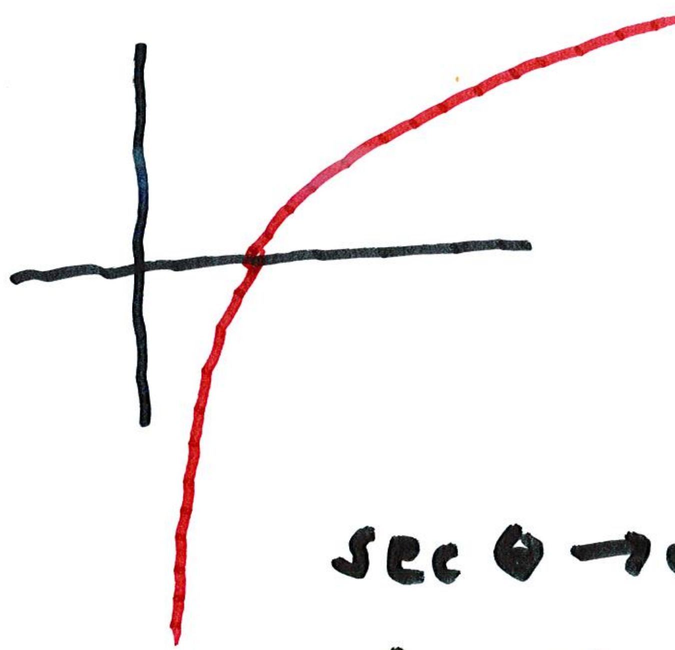
$$= \ln|\sec t| - \ln|\sec 0|$$

$$= \ln|\sec t|$$



As $t \to \frac{\pi}{2}$

$\sec t \to \infty$



As $x \to \infty$

$\ln x \to \infty$

$\therefore \cancel{\lim_{t \to \frac{\pi}{2}} \tan}$

$\sec \theta \to \infty$

$\ln|\sec \theta| \to \infty$

$\therefore$ As $t \to \frac{\pi}{2}$, $\lim\limits_{t \to \frac{\pi}{2}} \ln|\sec t|$

$= \infty$

$\therefore \displaystyle\int_0^{\frac{\pi}{2}} \tan\theta \, d\theta$ is divergent.