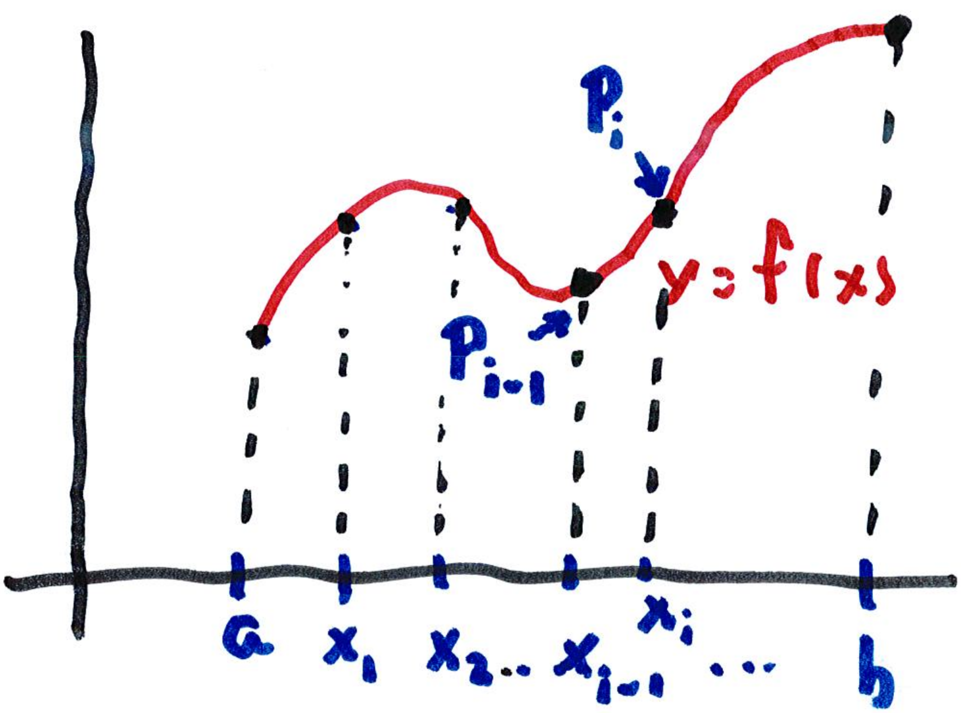
8.1 Arclength

Given a curve $y = f(x)$ for $a \le x \le b$, we want to find a formula for the length.



As usual, we define points

$P_i = (x_i, f(x_i))$, $i = 0, 1, \ldots, n$.

where $x_i = a + i\Delta x$ and

$$\Delta x = \frac{b-a}{n}$$

We define the **length of C**

by

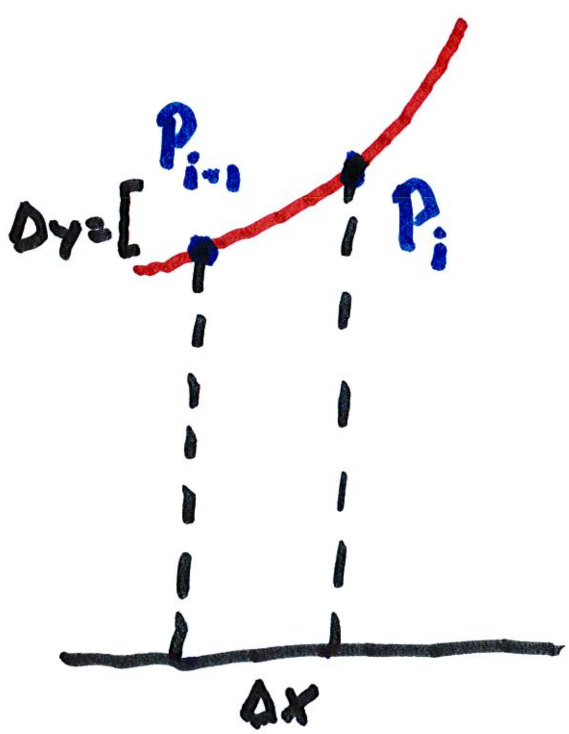
$$L = \lim_{n \to \infty} \sum_{i=1}^{n} |P_i - P_{i-1}|$$

$$|P_i - P_{i-1}| = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

where $\Delta y = f(x_i) - f(x_{i-1})$

Let m_i = slope of segment between P_{i-1} and P_i. Then

$$\Delta y \approx m_i \Delta x$$
$$\approx f'(x_i)\Delta x$$



$\therefore$

$$|P_i - P_{i-1}| \approx \sqrt{(\Delta x)^2 + (f'(x_i)\Delta x)^2}$$

$$= \sqrt{1 + (f'(x_i))^2}\ \Delta x$$

and so,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(x_i))^2}\ \Delta x$$

$$\boxed{L = \int_a^b \sqrt{1 + (f'(x))^2}\ dx}$$

One can also write

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

For most functions this is very hard or impossible to compute.

Ex. Find length of curve

$y = x^{3/2}$ for $0 \leq x \leq 2$.

$$f(x) = x^{3/2} \rightarrow f'(x) = \frac{3}{2} x^{1/2}$$

$$\int_0^2 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx$$

$$= \int_0^2 \sqrt{1 + \frac{9x}{4}}\, dx$$

Set $u = 1 + \frac{9x}{4} \rightarrow du = \frac{9}{4}dx$

$x = 0 \rightarrow u = 1$

or $dx = \frac{4}{9}du$

$x = 2 \rightarrow u = 1 + \frac{9}{2}$

$= \frac{11}{2}$

$$\int = \int_1^{11/2} \sqrt{u} \, \frac{4}{9}\, du$$

$$= \frac{2}{3} \cdot \frac{4}{9} \, u^{3/2} \Big|_1^{11/2}$$

$$= \frac{8}{27} \cdot \left(\frac{11}{2}\right)^{3/2} - \frac{8}{27}$$

$$= \frac{8}{27} \left(\left(\frac{11}{2}\right)^{3/2} - 1 \right)$$

Ex. Find length of curve C defined by $y = \frac{x^4}{8} + \frac{1}{4x^2}$,

$1 \le x \le 2$

$$\frac{dy}{dx} = \frac{4x^3}{8} - \frac{2}{4x^3} = \frac{x^3}{2} - \frac{1}{2x^3}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^6}{4} - \frac{1}{2} + \frac{1}{4x^6}$$

$$\rightarrow L = \int_1^2 \sqrt{\frac{x^6}{4} + \frac{1}{2} + \frac{1}{4x^6}}\, dx$$

$$= \int_1^2 \sqrt{\left(\frac{x^3}{2} + \frac{1}{2x^3}\right)^2}\, dx$$

$$= \int_1^2 \frac{x^3}{2} + \frac{1}{2x^3} \quad dx$$

$$= \frac{x^4}{8} - \frac{1}{4x^2} \Bigg|_1^2$$

$$= \left(2 - \frac{1}{16}\right) - \left(\frac{1}{8} - \frac{1}{4}\right)$$

$$= \cancel{\frac{27}{16}} = \frac{33}{16}$$

Ex. Find length of C defined by $y = 3 + \frac{1}{2}\cosh 2x$, $0 \le x \le 1$

$$\frac{dy}{dx} = \sinh 2x$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2 2x$$

$$\cosh^2 u - \sinh^2 u = 1$$

$$\rightarrow 1 + \sinh^2 u = \cosh^2 u$$

$$\therefore L = \int_0^1 \sqrt{1 + \sinh^2 2x}\ dx$$

$$= \int_0^1 \cosh 2x$$

$$= \frac{1}{2} \sinh 2x \Big|_0^1$$

$$= \frac{1}{2} \sinh(2) - 0$$

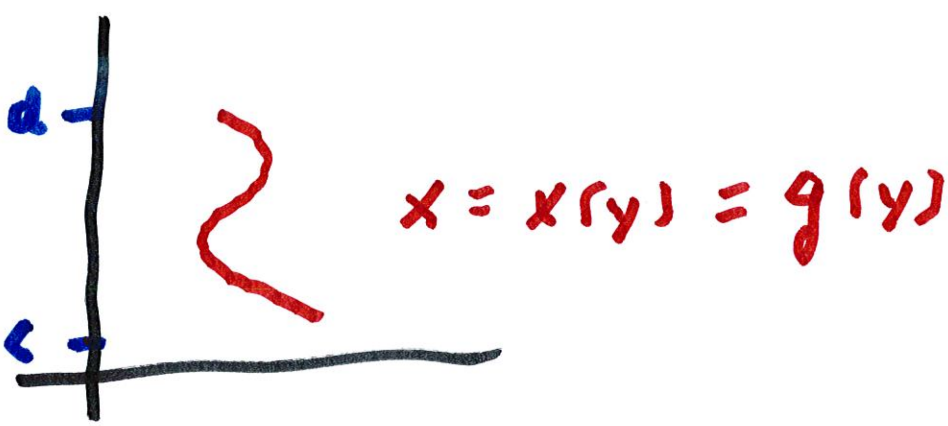
$$= \frac{1}{2} \sinh 2$$

If $x = x(y)$, then the same

$c \le y \le d$

method shows that

$$\text{Length of } C = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

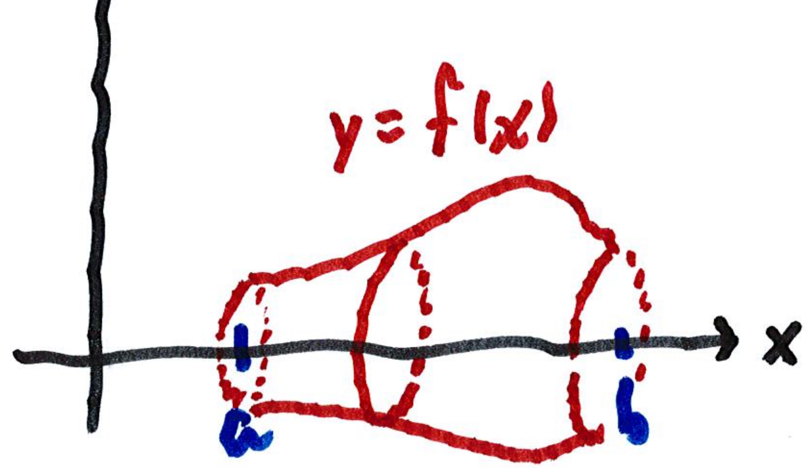


8.2 Area of a surface of revolution

Consider a curve C,

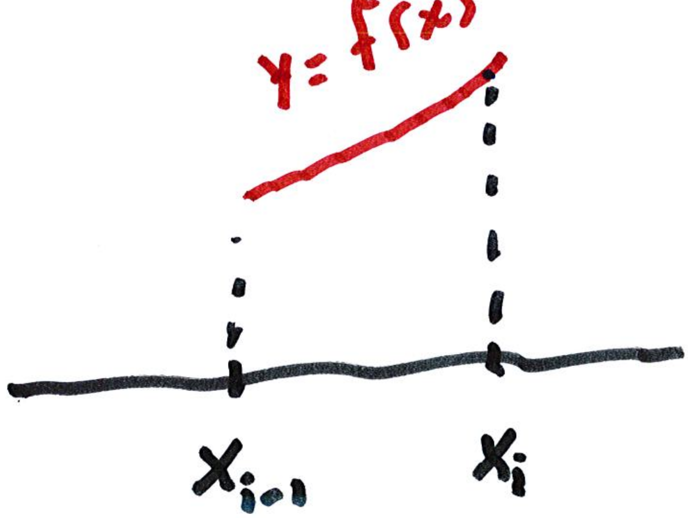
$y = f(x)$, $a \le x \le b$,

and rotate C about the x-axis. What is its surface area?



Consider a short segment of C.

$x_{i-1} \le x \le x_i$, $y = f(x)$.



We already showed

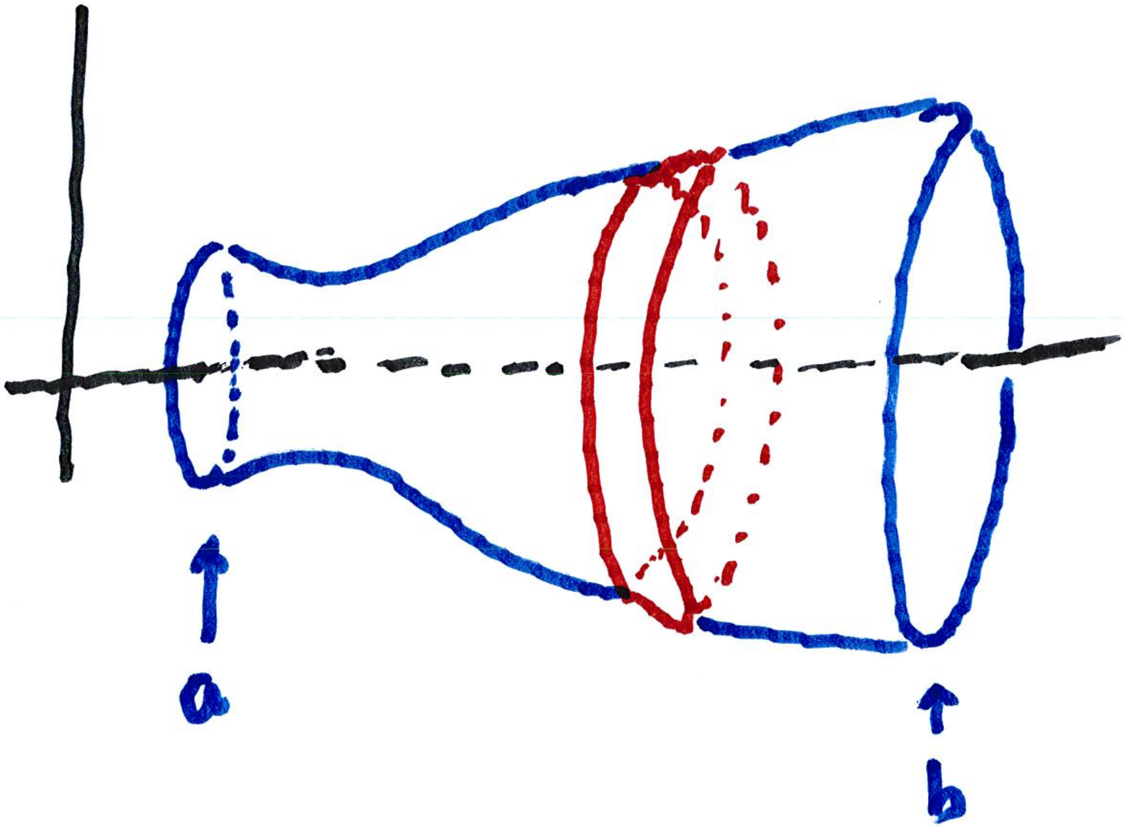
$$\Delta L = \Delta s \approx \sqrt{1 + \left(f'(x_i)\right)^2}\ \Delta x$$

If we rotate the segment about the x-axis, it generates a surface with surface area

$$\Delta S = 2\pi f(x_i)\sqrt{1 + \left(f'(x_i)\right)^2}\ \Delta x$$

Summing up over $i = 1, \ldots, n$, and letting $n \to \infty$, we get

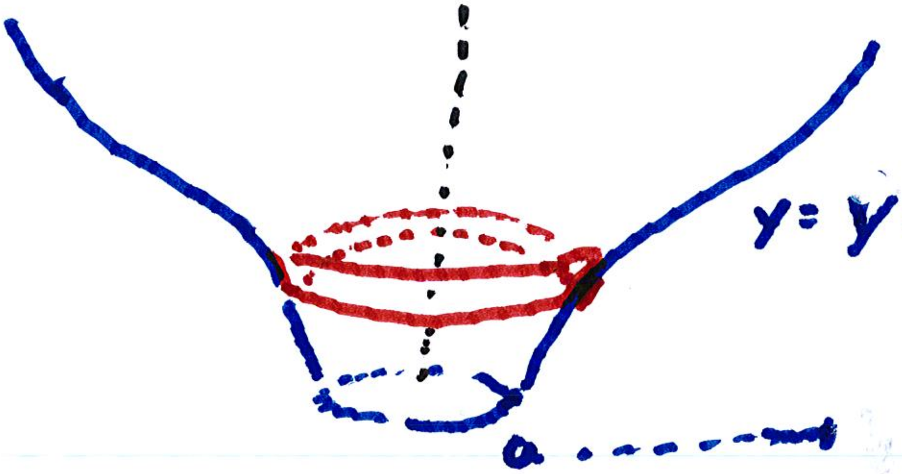
$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\ dx$$



or $$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad (1)$$

~~If [illegible], $c \le y \le d$~~

Around y-axis



$$S = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

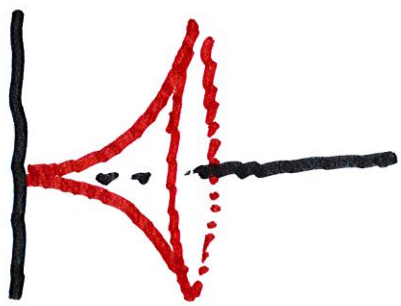
or if $x = x(y)$

$$S = \int_c^d 2\pi \underbrace{x}_{x = x(y)} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Ex. Rotate $y = x^3$ for

$0 \le x \le 1$ about the x-axis.

Find the surface area:



$y(x) = x^3$

$\frac{dy}{dx} = 3x^2$

$$S = \int_0^1 2\pi x^3 \sqrt{1 + 9x^4}\, dx$$

Set $u = 1 + 9x^4$

$\rightarrow du = 36x^3 dx \rightarrow x^3 dx = \frac{du}{36}$

$x = 0 \to u = 1 \qquad x = 1 \to u = 10$

$$S = \int_1^{10} 2\pi \sqrt{u} \, \frac{du}{36}$$

$$= \frac{\pi}{18} \cdot \frac{2}{3} u^{3/2} \Big|_1^{10}$$

$$= \frac{\pi}{27} \left(10^{3/2} - 1 \right)$$

If C is $x = x(y)$, $c \le y \le d$,

then

$$S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \quad (2)$$

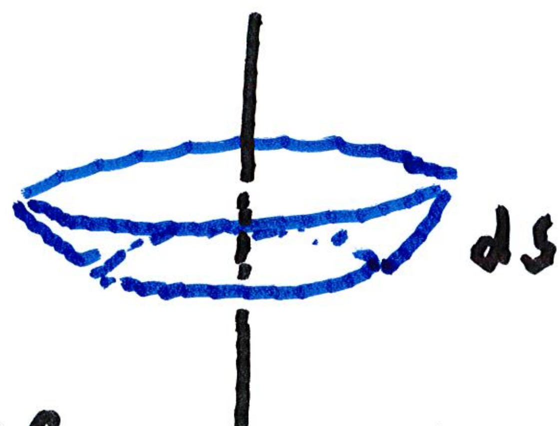
We can summarize (1) and (2)

by $S = \int 2\pi y \, ds$

If we rotate C about the

y-axis: $S = \int 2\pi x \, ds$

The curve $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y-axis. Find the surface area



$$dS = 2\pi x \, ds$$

$$= 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$y = x^2$

$\frac{dy}{dx} = 2x$

$$\therefore S = \int_1^2 2\pi x \sqrt{1 + (2x)^2} \, dx$$

$$S = 2\pi \int_1^2 x\sqrt{1+4x^2}\, dx$$

Set $u = 1 + 4x^2 \rightarrow du = 8x\, dx$

or $x\, dx = \dfrac{du}{8}$

$x = 1 \rightarrow u = 5$

$x = 2 \rightarrow u = 17$

$$S = 2\pi \int_5^{17} \sqrt{u}\, \frac{du}{8}$$

Ex. The curve $y = x^2$, $1 \le x \le 2$

is rotated about the y-axis.

Find S.

$$S = \int_1^2 2\pi x \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^2 2\pi x \sqrt{1 + 4x^2}\, dx$$

$$= \frac{\pi}{4} \int_1^2 8x\sqrt{1+4x^2}\ dx$$

$u = 1+4x^2 \quad du = 8x\,dx$

$x=1 \rightarrow u=5$

$x=2 \rightarrow u=17$

$$= \frac{\pi}{4} \int_5^{17} \sqrt{u}\ du$$

$$= \frac{\pi}{4} \cdot \frac{2}{3}\, u^{3/2} \Big|_5^{17} = \underline{\underline{\frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)}}$$

$$= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_{5}^{17}$$

$$= \frac{\pi}{6} \left(17^{3/2} - 5^{3/2} \right)$$

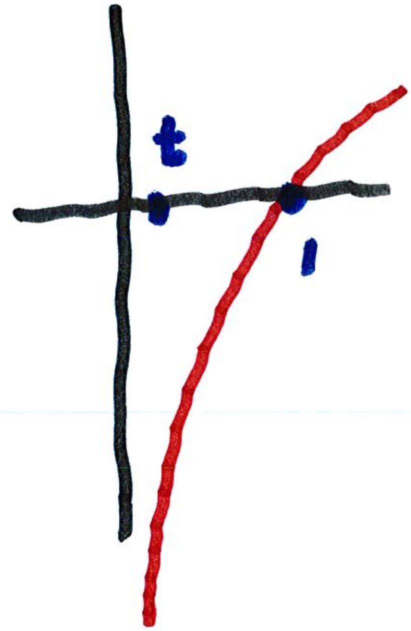
Some improper integrals:

$$\int_0^1 \frac{\ln x}{x}\, dx \qquad \left| \frac{\ln x}{x} \right| \to \infty$$

as $x \to 0$

Choose t with $0 < t < 1$

$$\int_t^1 \frac{\ln x}{x}\,dx$$



Set $u = \ln x$

$du = \dfrac{dx}{x}$

$x = 1 \rightarrow u = \ln 1 = 0$

$x = t \rightarrow u = \ln t$

$$= \int_{\ln t}^{0} u\,du = \frac{u^2}{2}\Big|_{\ln t}^{0}$$

$$= 0 - \frac{(\ln t)^2}{2} = \frac{-(\ln t)^2}{2}$$

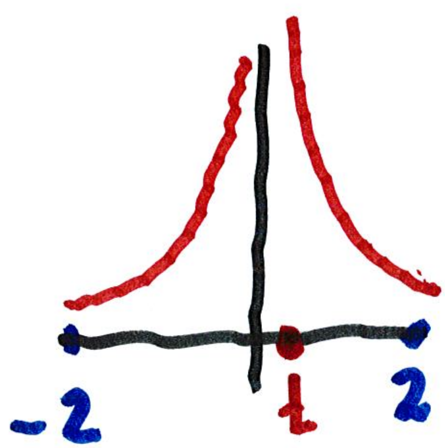
As $t \to 0^+$, $\ln t \to -\infty$

$\Rightarrow (\ln t)^2 \to +\infty$

$\Rightarrow \dfrac{-(\ln t)^2}{2} \to -\infty.$

$\therefore$ Integral Diverges

Ex. Is $\displaystyle\int_{-2}^{2} \frac{dx}{x^6}$ convergent?



Choose t

$0 < t < 1$