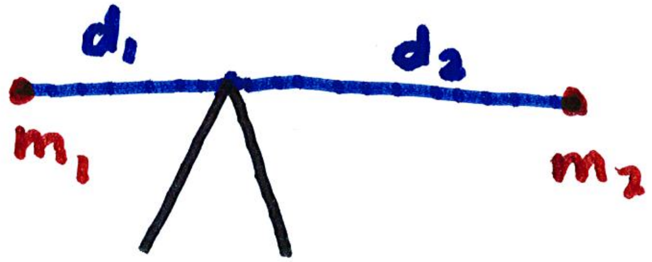
# 8.3 Moments and Centers of Mass.

Consider a rod with masses $m_1$ and $m_2$ on opposite sides of a fulcrum so that the masses are of distance $d_1$ and $d_2$ from the fulcrum.




The fulcrum is the <u>center of mass.</u>

The rod will balance if

$$m_1 d_1 = m_2 d_2$$

Now suppose the rod lies on the x-axis with $m_1$ at $x_1$ and $m_2$ at $x_2$, and with the center of mass at $\bar{x}$. Then

$$d_1 = \bar{x} - x_1 \quad \text{and} \quad d_2 = x_2 - \bar{x}$$

and

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$m_1\bar{x} + m_2\bar{x} = m_1x_1 + m_2x_2$$

$$\Rightarrow \quad \bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

The numbers $m_1X_1$ and $m_2X_2$ are moments of the masses $m_1$ and $m_2$. Note that $m_1 + m_2$ is the total mass.

In general, if we have a system of n particles with masses $m_1, m_2, \dots, m_n$

located at $x_1, x_2, \dots, x_n$ on

the x-axis, the center of

m is at

ass

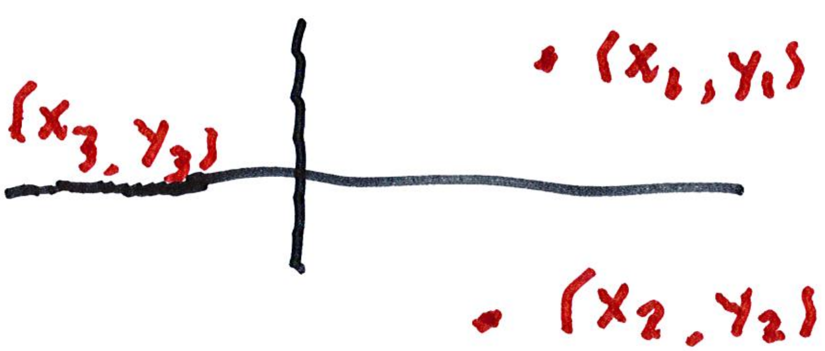
$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i x_i}{m}$$

where $m = \sum m_i$ is the total

mass.

We let $M = \sum_{i=1}^{n} m_i x_i$ be the moment of the system about the origin. $\therefore \bar{x} = \frac{M}{m}$

Now consider a system of n particles with masses at $(x_1, y_1), \ldots, (x_n, y_n)$



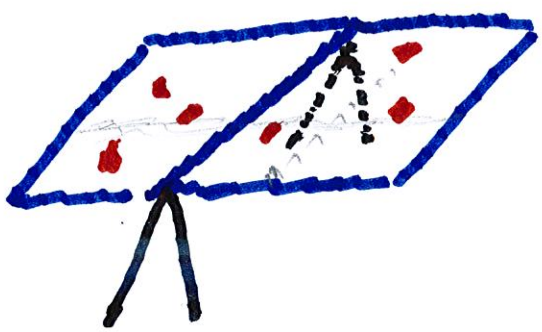
We define the moment of the system about the y-axis to be

$$M_y = \sum_{i=1}^{n} m_i x_i$$

and the moment about the x-axis to be

$$M_x = \sum_{i=1}^{n} m_i y_i$$

$M_y$ measures the tendency of the system to rotate about the y-axis. Then $M_x$ measures the tendency of the system to rotate about the x-axis.



The center of mass of the system is at $(\bar{x}, \bar{y})$ where

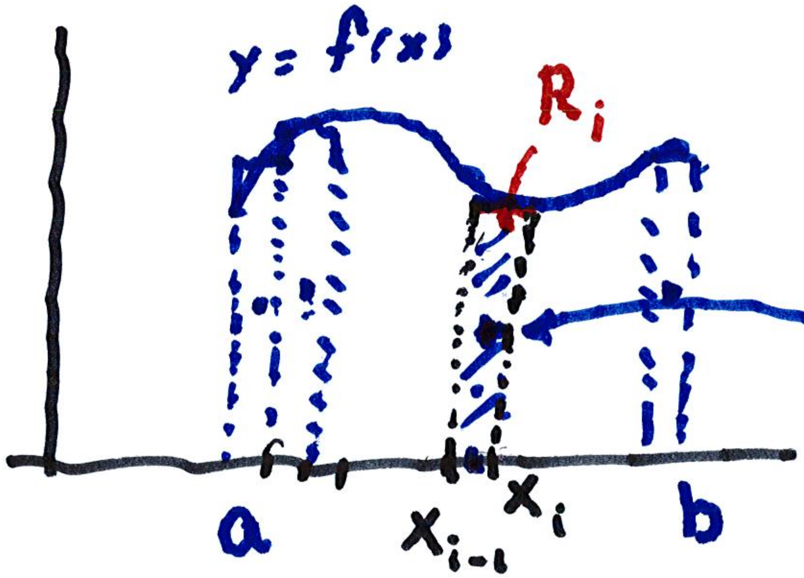
$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}.$$

Where $m = \sum_{i=1}^{n} m_i$ = total mass.

Now consider a flat plate

(a lamina) $\quad 0 < y < f(x)$

$\qquad\qquad\qquad a < x < b$




Subdivide

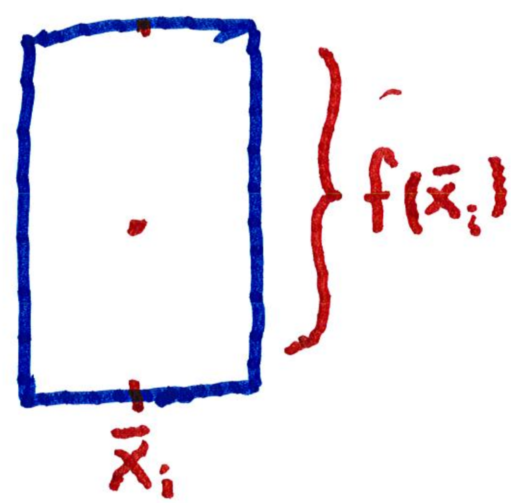
$$\left(\bar{x}_i, \frac{f(\bar{x}_i)}{2}\right)$$

Rectangle defined

$$x_{i-1} \le x \le x_i \qquad 0 \le y \le f(\bar{x}_i)$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$. Note that the center of mass of $R_i$

$$= \left( \bar{x}_i, \frac{f(\bar{x}_i)}{2} \right)$$




Mass of $R_i$

$$= \rho f(\bar{x}_i) \Delta x$$

We first compute the moment $M_y(R_i)$:

$$M_y^i = \underbrace{\rho f(\bar{x}_i)\Delta x}_{\text{mass}} \cdot \underbrace{\bar{x}_i}_{\text{dist. of } (\bar{x}_i, f(\bar{x}_i))}$$

$$= \rho \bar{x}_i f(\bar{x}_i) \Delta x$$

Summing over i,

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i) \Delta x$$

$$= \rho \int_a^b x f(x)\, dx$$

Similarly

$$M_x(R_i) = \left[\rho f(\bar{x}_i)\,\Delta x\right] \cdot \frac{1}{2} f(\bar{x}_i)$$

$$= \rho \cdot \frac{1}{2}\left(f(\bar{x}_i)\right)^2 \Delta x$$

Summing over $i$ :

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \cdot \frac{1}{2}\left(f(\bar{x}_i)\right)^2 \Delta x$$

$$= \rho \int_a^b \frac{1}{2}\left(f(x)\right)^2 dx$$

Since $m\bar{x} = M_y$, and $m\bar{y} = M_x$

$\longrightarrow$ $\bar{x} = \dfrac{M_y}{m}$ and $\bar{y} = \dfrac{M_x}{m}$

where $m = \rho A = \rho \int_a^b f(x)\,dx$

Hence,

$$\bar{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x f(x)\,dx}{\rho \int_a^b f(x)\,dx} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2}\{f(x)\}^2\,dx}{\rho \int_a^b f(x)\,dx}$$

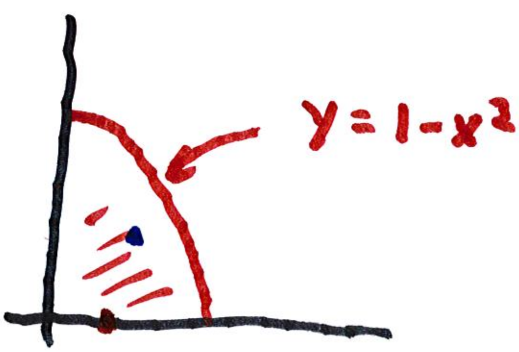
In summary

$$\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\{f(x)\}^2\,dx$$

where $A$ = area of plate

Ex Let $D$ = region in first quadrant bounded by $y = 0$, $x = 0$ and $y = 1 - x^2$. Find the center of mass




$$M_y = \int_0^1 x(1-x^2)\,dx$$

$$= \frac{x^2}{2} - \frac{x^4}{4} \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{4} = \underline{\underline{\frac{1}{4}}}$$

$$M_x = \int_0^1 \frac{1}{2}(1-x^2)^2\,dx$$

$$= \frac{1}{2}\int_0^1 (1-2x^2+x^4)\,dx$$

$$= \frac{1}{2}\left(x - \frac{2x^3}{3} + \frac{x^5}{5}\right)\Big|_0^1$$

$$= \frac{1}{2}\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{1}{2}\cdot\frac{8}{15} = \frac{4}{15}$$

Area $A = \int_0^1 (1-x^2)\,dx$

$$= x - \frac{x^3}{3}\Big|_0^1 = \frac{2}{3}$$

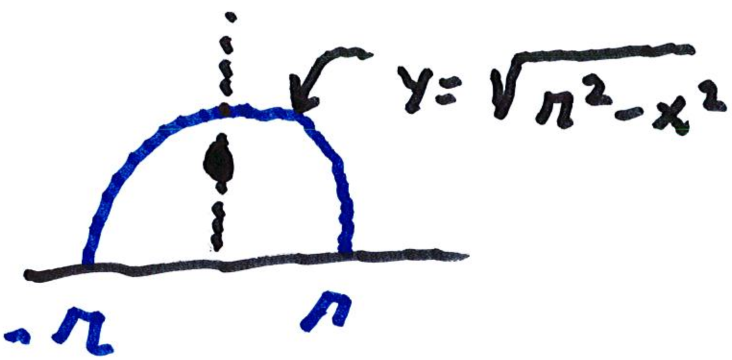
$$\therefore \quad \bar{x} = \frac{\frac{1}{4}}{\frac{2}{3}} = \frac{3}{8}$$

and $$\bar{y} = \frac{4/15}{\frac{2}{3}} = \frac{2}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{3}{8}, \frac{2}{5}\right)$$

Find the centroid $(\bar{x}, \bar{y})$ of

$D$ bounded by $y = 0$ and $y = \sqrt{r^2 - x^2}$




$$M_y = \int_{-r}^{r} x\sqrt{r^2 - x^2}\, dx = 0$$

because $x\sqrt{r^2 - x^2}$ is odd

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$$M_x = \frac{1}{2}\int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx$$

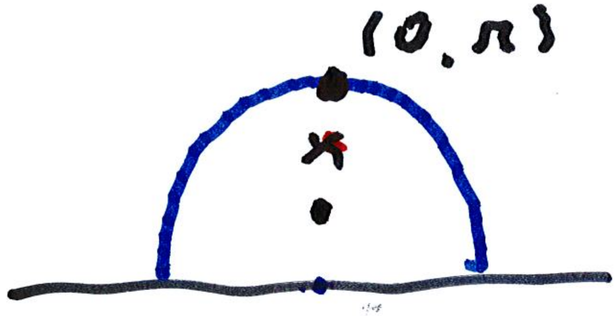
$$= \frac{2}{2}\int_{0}^{r} r^2 - x^2\, dx$$

$$= r^2x - \frac{x^3}{3}\Big|_0^r = r^3 - \frac{r^3}{3}$$

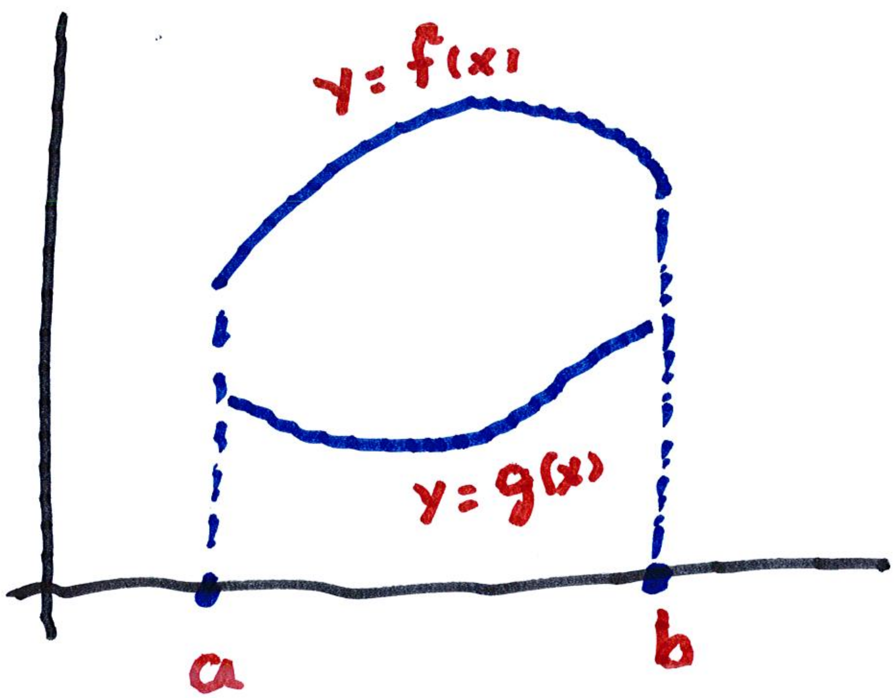
$$= \frac{2r^3}{3}$$

$$\text{Area} = \frac{\pi r^2}{2}$$

$$\therefore \bar{y} = \frac{\frac{2r^3}{3}}{\frac{\pi r^2}{2}} = \frac{4r}{3\pi}$$



Now suppose $D = \left\{ (x,y) ;\ \begin{array}{c} a \le x \le b \\ \text{and} \\ g(x) \le y \le f(x) \end{array} \right.$

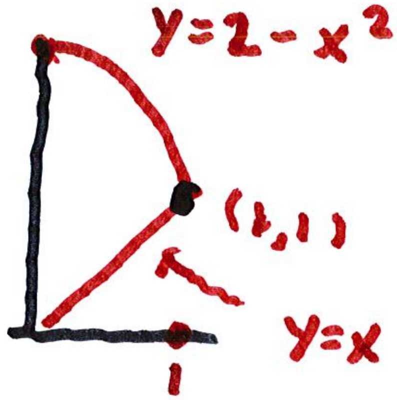



Then $M_y = \displaystyle\int_a^b x \{ f(x) - g(x) \} \, dx$

and $M_x = \displaystyle\int_a^b \frac{1}{2} \left( f(x) \right)^2 - \frac{1}{2} \{ g(x) \}^2 \, dx$

Ex. Find $(\bar{x}, \bar{y})$ for

$$D = \left\{ (x, y);\ 0 \le x \le 1 \text{ and } x \le y \le 2 - x^2 \right\}$$




$$M_y = \int_0^1 x\,(2 - x^2 - x)\,dx$$

$$= x^2 - \frac{x^4}{4} - \frac{x^3}{3} \Big|_0^1$$

$$= 1 - \frac{1}{4} - \frac{1}{3} = \frac{5}{12}$$

$$M_x = \frac{1}{2}\int_0^1 \left((2-x^2)^2 - x^2\right) dx$$

$$= \frac{1}{2}\int_0^1 4 - 4x^2 + x^4 - x^2 \, dx$$

$$= \frac{1}{2}\left(4x - \frac{4x^3}{3} + \frac{x^5}{5} - \frac{x^3}{3}\right)\Big|_0^1$$

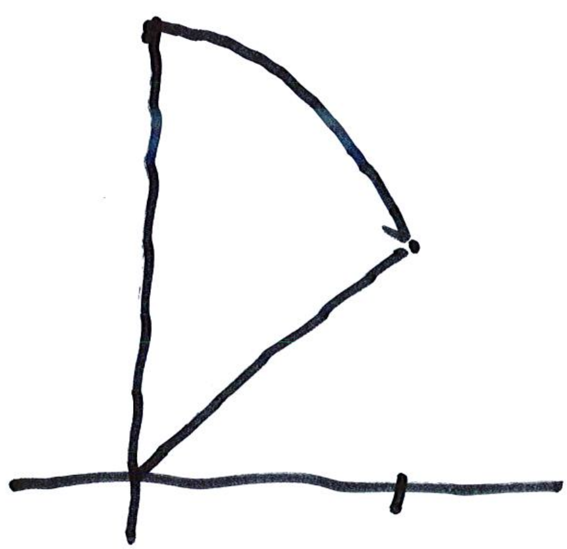
$$= \frac{1}{2}\left(4 - \frac{4}{3} + \frac{1}{5} - \frac{1}{3}\right) = \frac{19}{15}$$

Area $A = \int_0^1 2 - x^2 - x \, dx$

$$= \left(2x - \frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_0^1 = \frac{7}{6}$$

$$\therefore \bar{x} = \frac{M_y}{A} = \frac{\frac{5}{12}}{\frac{7}{6}} = \frac{5}{14}$$

$$\text{and } \bar{y} = \frac{M_x}{A} = \frac{\frac{19}{15}}{\frac{7}{6}} = \frac{114}{105}$$



$\bar{x} < \frac{1}{2}$

Then $M_y = \int_a^b \rho f(x)\, x \, dx$

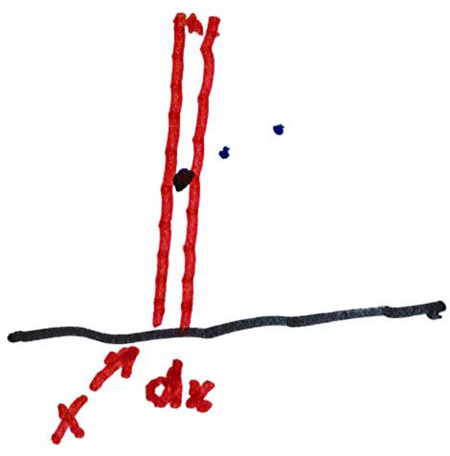
Note that $\rho f(x)\, dx$ =

mass of an infinitesimal strip at $x$,

and $x$ = distance of strip from

the $y$-axis.

$$M_x = \int_a^b f(x) \cdot \frac{f(x)}{2}\, dx$$




If the thin strip is concentrated at $\left(x, \frac{f(x)}{2}\right)$, then

$\frac{f(x)}{2}$ = distance from $x$-axis