11.1 Sequences

A sequence is a list of numbers written in a definite order.

\[ a_1, a_2, a_3, \ldots, a_n, \ldots \]

We sometimes write

\[ \{a_n\} \text{ or } \{a_n\}_{n=1}^{\infty} \]

Sometimes we give a formula for the \( n \)-th term.
Sometimes we list the first several terms

\[ \left\{ 2^{-n} \right\}_{n=1}^{\infty}, \quad a_n = 2^{-n} \] \[ \left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \right\} \]

\[ \left\{ \frac{n}{n-2} \right\}_{n=3}^{\infty}, \quad a_n = \frac{n}{n-2} \]

\[ \left\{ \frac{3}{1}, \frac{4}{2}, \frac{5}{3}, \ldots, \frac{n}{n-2}, \ldots \right\} \]

Sometimes it's hard to find a formula:
\{ \frac{-2}{2}, \frac{4}{3}, \frac{-8}{4}, \frac{16}{5}, \ldots \} \\
2, 4, 8, 16 \quad 2^n \\
2, 3, 4, 5 \quad n+1 \\
\alpha_n = \frac{2^n}{n+1} \quad \text{Need - signs:} \\
\alpha_n = (-1)^n \frac{2^n}{n+1}
Sometimes \( \{a_n^2\} \) is defined recursively:

Set \( a_1 = 1 \) and \( a_n = 2a_{n-1} + n^2 \) when \( n \geq 2 \).

\[ \{1, 6, 21, 58, 141, \ldots \} \]

Finding a formula \( f(n) \) for the \( n \)-th term seems very hard.
We want to define the limit of a sequence.

We want to say that

\[ \lim_{n \to \infty} a_n = L \]

if the \( n \)-th term gets closer and closer to \( L \) as \( n \) increases.

But if \( a_n = (1 + (-1))^n \) and \( L = 2 \) then \( a_n = 2 \) if \( n \) is even.

Is this good enough?
Precise Definition of $\lim_{n \to \infty} a_n$.

We say $\lim_{n \to \infty} a_n = L$ if for every number $\epsilon > 0$ there is an integer $N_\epsilon > 0$ such that if $n > N_\epsilon$, then $|a_n - L| < \epsilon$. In this case, we say $\{a_n\}$ is convergent.
In other words, if $n$ is large enough, say $n > N_\varepsilon$, then $a_n$ is close to $L$, i.e. $|a_n - L| < \varepsilon$.

In this case it seems that $N_\varepsilon = 3$ works.

$1 < a_n - L < 3$}

$-3 < a_n - L < 3 \implies L - 3 < a_n < L + 3$
If $n > N$, then $|a_n - L| < \varepsilon$

**Thm:** If $\lim_{x \to \infty} f(x) = L$,

and $a_n = f(n)$, then

$$\lim_{n \to \infty} a_n = L$$
Def'n: \( \lim_{n \to \infty} a_n = \infty \) means that for every positive number \( M \), there is an integer \( N \) so that

\[
\text{if } n > N \text{ then } a_n > M
\]

Such a sequence is divergent, but still in a useful way.
Limit Laws.

Suppose \( \{a_n\} \) and \( \{b_n\} \) are convergent sequences. Then

1. \( \lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n \)

2. \( \lim_{n \to \infty} (a_n - b_n) = \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n \)

3. \( \lim_{n \to \infty} (a_n \cdot b_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n \)
5. \( \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n} \)

(if \( \lim b_n \neq 0 \))

6. \( \lim_{n \to \infty} a_n^p = \left( \lim_{n \to \infty} a_n \right)^p \) if \( p > 0 \) and \( a_n > 0 \).

Ex. Show \( \lim_{n \to \infty} \frac{2n+1}{3n+2} = \frac{2}{3} \)

\[
\frac{2n+1}{3n+2} = \frac{n \left( 2 + \frac{1}{n} \right)}{n \left( 3 + \frac{2}{n} \right)} = \frac{2 + \frac{1}{n}}{3 + \frac{2}{n}}
\]
For functions, we know that

$$\lim_{x \to \infty} \frac{1}{x^n} = 0 \text{ if } n > 0$$

$\therefore \lim_{n \to \infty} \frac{1}{n^2} = 0 \text{ if } n > 0 \quad \text{(by 6)}$

$\therefore \lim_{n \to \infty} \frac{1}{n} = 0 \quad \text{(if } n = 1 \text{)}$

$\therefore \lim_{n \to \infty} 2 + \frac{1}{n} = 2 + 0 = 2 \quad \text{(by 1)}$

and $\lim_{n \to \infty} \frac{2}{n} = 2 \cdot 0 = 0 \quad \text{(by 3)}$
\[ \lim_{n \to \infty} \left(3 + \frac{2}{n}\right) = 3 + 0 = 3 \quad \text{(by 1)} \]

\[ \lim_{n \to \infty} \frac{2 + \frac{1}{n}}{3 + \frac{2}{n}} = \frac{2}{3} \quad \text{(by 5)} \]

Thm. If \( \lim_{n \to \infty} a_n = L \geq 0 \)

and \( \lim_{n \to \infty} b_n = \infty \), then

\[ \lim_{n \to \infty} a_n b_n = \infty \]

Ex. \( d_n = \sqrt{4n^2 + n} = n\sqrt{4 + \frac{1}{n}} \)
Ex. Show that \( \lim_{{n \to \infty}} \frac{n}{\sqrt{2n+7}} = 0 \)

Note that \( \frac{n}{\sqrt{2n+7}} = \frac{\sqrt{n} \cdot \sqrt{n}}{\sqrt{2n+7}} \)

\[= \sqrt{n} \cdot \sqrt{\frac{n}{{2n+7}}} \to \sqrt{\frac{1}{2}} > 0 \]

\[= \sqrt{n} \cdot \sqrt{\frac{1}{{2+\frac{7}{n}}}} \to \infty \]
\[
\lim_{n \to \infty} \sqrt{n} = \infty
\]

and \[
\lim_{n \to \infty} \sqrt{\frac{1}{2 + \frac{2}{n}}} = \sqrt{\frac{1}{2}} > 0
\]

\[
\therefore \lim_{n \to \infty} \sqrt{n} \cdot \sqrt{\frac{1}{2 + \frac{2}{n}}} = \infty
\]

**Ex.** Show \[
\lim_{n \to \infty} \frac{\ln n}{n} = 0
\]

**L'Hôpital's Rule** \[
\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0
\]

Hence, \[
\lim_{n \to \infty} \frac{\ln n}{n} = 0
\]
Show \( \lim_{n \to \infty} \frac{n^2}{2^n} = 0 \)

\[
\lim_{x \to \infty} \frac{x^2}{2^x} = \lim_{x \to \infty} \frac{2x}{(\ln 2)^2 2^x}
\]

\[
= \lim_{x \to \infty} \frac{2}{(\ln 2)^2 2^x} = 0
\]

\[
= \lim_{n \to \infty} \frac{n^2}{2^n} = 0
\]
\[ \lim_{x \to a} \frac{1}{x} = 0. \]
\[ \lim_{n \to \infty} \ln n = 0. \]

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**Squeeze Thm:**

Suppose \( a_n \leq b_n \leq c_n \)

for all \( n \geq n_0 \), and that

\[ \lim_{n \to \infty} a_n = L \quad \text{and} \quad \lim_{n \to \infty} c_n = L \]

Then \( \lim_{n \to \infty} b_n = L \)
Ex. Let \( C_n = \frac{n}{3n^2 + 2} \)

\[
= \frac{n}{n^2 \left(3 + \frac{2}{n^2}\right)} = \frac{1}{n} \cdot \frac{1}{3 + \frac{2}{n^2}}
\]

\[
\downarrow \quad \downarrow \quad \downarrow \\
0 \quad \frac{1}{3}
\]

\[
\lim_{n \to \infty} \frac{n}{3n^2 + 2} = 0
\]

Let \( b_n = (-1)^n \cdot \frac{n}{3n^2 + 2} \)

and \( a_n = -\frac{n}{3n^2 + 2} \rightarrow -0 = 0 \)
\[-\frac{n}{3n^2+2} \leq (-1)^n \cdot \frac{n}{3n^2+2} \leq \frac{n}{3n^2+2}\]

\[a_n \downarrow 0\]

\[\lim_{n \to \infty} \frac{(-1)^n n}{3n^2+2} = 0\]
\[-\frac{\sqrt{n}}{2n+3} \leq \frac{(-1)^n \sqrt{n}}{2n+3} \leq \frac{\sqrt{n}}{2n+3}\]

\[\lim_{{n \to \infty}} \frac{(-1)^n \sqrt{n}}{2n+3} = 0 \text{ too.}\]

Then suppose \(\lim_{{n \to \infty}} a_n = L\) and that \(f\) is continuous.
at \( L \). Then

\[
\lim_{n \to \infty} f(x_n) = f(L)
\]

Ex. Compute \( \lim_{n \to \infty} \tan^{-1}\left( \frac{n+2}{n+4} \right) \)

Set \( x_n = \frac{n+2}{n+4} = 1 + \frac{2}{n+4} \to \frac{1}{1} = 1 \)

Also \( \tan^{-1} x \) is continuous at \( x = 1 \)

\[
\Rightarrow \lim_{x \to 1} \tan^{-1} x = \tan^{-1} 1 = \frac{\pi}{4}
\]
\[
:\lim_{n \to \infty} \tan^{-1}(\tan n) = \tan^{-1}1 = \frac{\pi}{2}
\]

Two useful limits:

If \( r > 1 \), then \( \lim_{n \to \infty} r^n = \infty \)

Ex. \( \lim_{n \to \infty} 2^n = \infty \), \( \lim_{n \to \infty} (1.1)^n = \infty \)

If \( 0 < r < 1 \), then \( \lim_{n \to \infty} r^n = 0 \)
Ex. \( \lim_{n \to \infty} \left( \frac{1}{3} \right)^n = 0 \)

and \( \lim_{n \to \infty} \left( 0.99 \right)^n = 0 \)

If \(-1 < r < 0\),

\( \lim_{n \to \infty} r^n = 0 \)

Ex. \( \lim_{n \to \infty} \left( -\frac{1}{2} \right)^n = 0 \)
Def'n. A sequence \( \{a_n\} \) is called **increasing** if \( a_n < a_{n+1} \) for all \( n \geq 1 \).

Similarly, \( \{a_n\} \) is called **decreasing** if \( a_n > a_{n+1} \) for all \( n \geq 1 \).

Ex. Show \( \alpha_n = 2 - \frac{3}{n} \) is inc. (If \( \{a_n\} \) is either increasing or decreasing, then \( \{a_n\} \) is monotonic.)
Set \( f(x) = 2 - \frac{3}{x} \)

\[ f'(x) = \frac{3}{x^2} > 0 \text{ if } x > 0 \]

\[ \therefore \text{ if } x_1 < x_2, \text{ then } f(x_1) < f(x_2) \]

Hence, \( f(n) < f(n+1) \) if \( n \geq 0 \)

or \( 2 - \frac{3}{n} < 2 - \frac{3}{n+1} \)

\[ \therefore a_n = 2 - \frac{3}{n} \text{ is inc.} \]
Def'n. We say \{a_n\} is bounded above if there is a number M so that
\[ a_n \leq M, \text{ for all } n. \]

Similarly, \{a_n\} is bounded below if \[ m \leq a_n, \text{ for all } n \geq 1. \]
Monotonic Seq. Thm.

If \( \{a_n\} \) is inc. and bounded above, then there is a number \( L \) so \( \lim_{n \to \infty} a_n = L \)