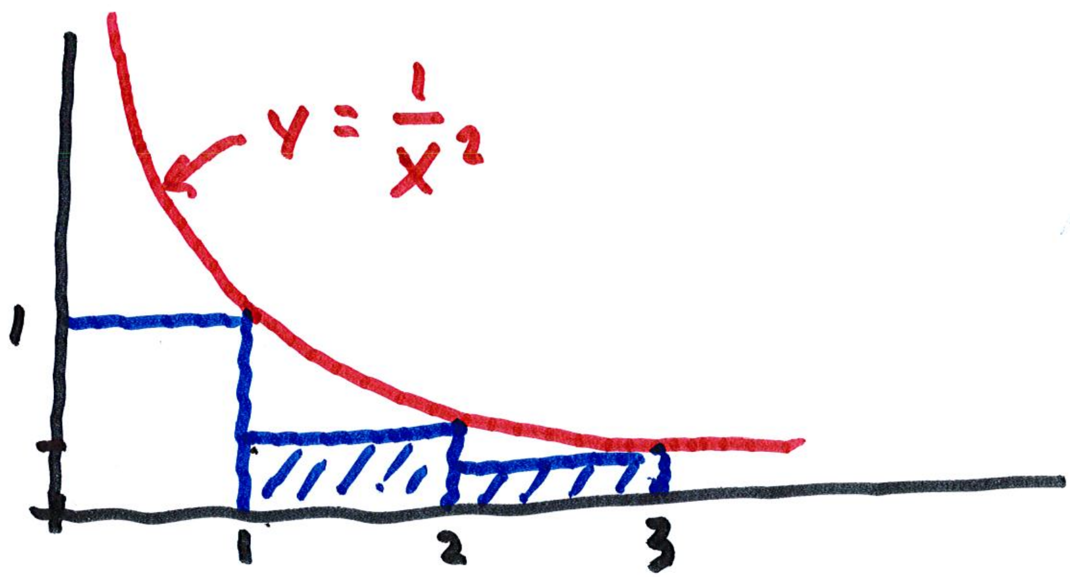
11.3 The Integral Test.

Does $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge?



Area = $\frac{1}{1^2}$ $\frac{1}{2^2}$ $\frac{1}{3^2}$

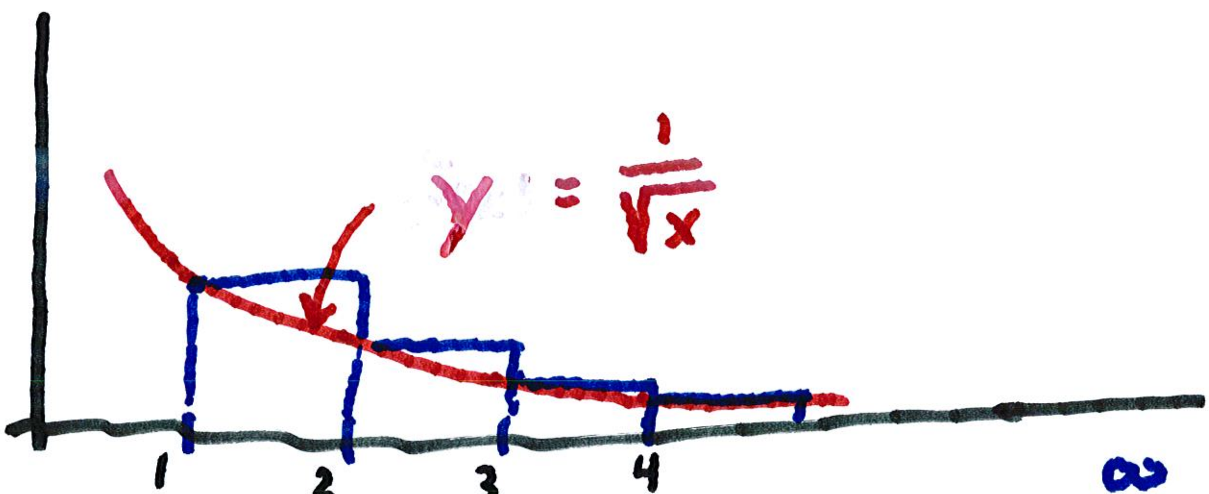
Area under $y = \frac{1}{x^2}$, for $1 \leq x < \infty$

$$= \int_1^{\infty} \frac{1}{x^2}\, dx$$

$$\therefore \sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{dx}{x^2} = 1$$

$$\rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + 1 = 2$$

Ex. Does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converge?



Area = $\frac{1}{\sqrt{1}} \quad \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{4}}$ $\qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$\parallel$

Total Area

Note that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{dx}{\sqrt{x}} = \infty$$

(since $p = \frac{1}{2} < 1$)

$$\therefore \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges}$$

<u>Integral Test</u>. Suppose $f(x)$ is a continuous and decreasing function on $[1, \infty)$.

Set $a_n = f(n)$. Then

(i) If $\int_1^\infty f(x)\,dx$ is convergent,

then $\sum_{n=1}^{\infty} a_n$ is convergent

and

(ii) If $\int_1^\infty f(x)\,dx$ is divergent

then $\sum_{n=1}^{\infty} a_n$ is divergent.

Recall $\int_1^\infty \frac{dx}{x^p}$ converges

if $p>1$ and diverges if $p \leq 1$.

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1$$

and diverges if $p \leq 1$.

This is called the <u>p-test</u>

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges.

$(p = \frac{2}{3} < 1)$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converges since $p = 1.1 > 1$.

Ex. Does $\sum_{n=1}^{\infty} \frac{1}{2n+5}$ converge?

$$\int_1^{\infty} \frac{dx}{2x+5} = \frac{1}{2}\int_1^{\infty} \frac{2\,dx}{2x+5}$$

$u = 2x+5$

$du = 2\,dx$

$$= \frac{1}{2}\int_7^{\infty} \frac{du}{u}$$

$$= \frac{1}{2} \ln u \Big|_7^{\infty} = \frac{1}{2}\ln \infty - \frac{1}{2}\ln 7$$

$$= \infty - \frac{1}{2}\ln 7 = \infty$$

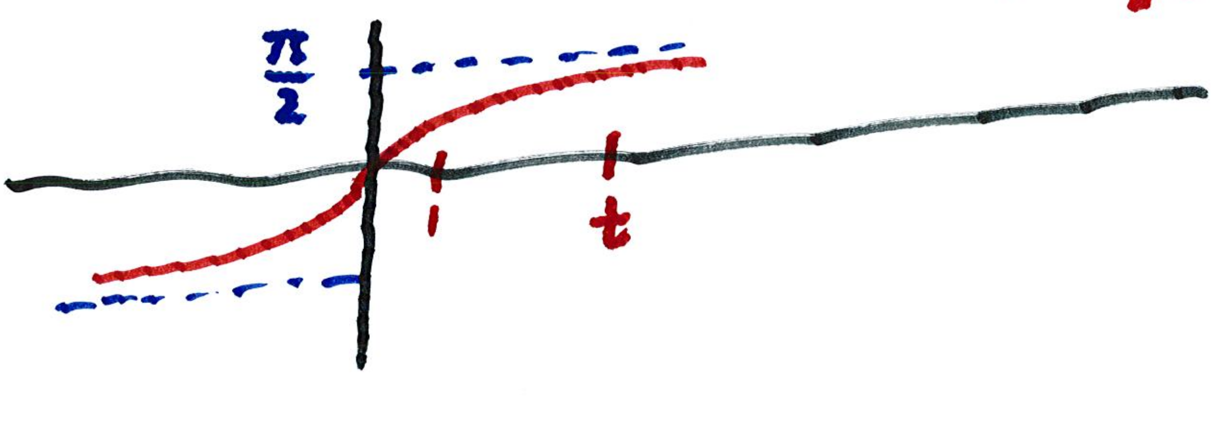
Some examples:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1} \rightsquigarrow \int_1^{\infty} \frac{dx}{x^2+1}$

$$\lim_{t\to\infty} \int_1^t \frac{dx}{x^2+1} = \lim_{t\to\infty} \tan^{-1}x \Big|_1^t$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \quad \therefore \sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

converges.



Ex. Look at $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

$f(x) = \frac{1}{x \ln x}$. One can show

$$f'(x) = \frac{-(\ln x + 1)}{(x \ln x)^2} < 0$$

$\therefore$ $f(x)$ is decreasing.

Look at $\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{t \to \infty} \int_2^{t} \frac{dx}{x \ln x}$

$u = \ln x \quad du = \frac{dx}{x}$

$$= \lim \int_{\ln 2}^{\ln t} \frac{du}{u} = \ln u \Big|_{\ln 2}^{\ln t}$$

$$= \ln(\ln t) - \ln(\ln 2) \to \infty$$

as $t \to \infty$

Hence $\displaystyle\int_2^\infty \frac{dx}{x \ln x}$ diverges

$\to$ $\displaystyle\sum_{n=2}^\infty \frac{1}{n \ln n}$ diverges

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Ex. Show $\displaystyle\sum_{n=0}^\infty e^{-n^2} n$

$\to$ $\displaystyle\int_0^\infty e^{-x^2} x \, dx = \frac{1}{2} \int_0^\infty e^{-u} \, du$

$u = x^2 \quad du = 2x \, dx$

$$= \frac{1}{2} < \infty$$

$\therefore$ Series Conv.