11.4 The Comparison Test

We know that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges

and $= 1$.

Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$.

For each n, we have

$$\frac{1}{2^n+1} < \frac{1}{2^n}$$

$\therefore$ It must be that

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1} < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Thus, it follows that

$$\sum_{n=1}^{\infty} \frac{1}{2^n+1} \text{ is } \underline{\text{convergent}}.$$

Now look at $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n}$.

For each $n = 1, 2, \ldots$, we have

$$\sqrt{n+1} > 1.$$

Hence $\displaystyle\frac{\sqrt{n+1}}{n} > \frac{1}{n}$. (*)

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, This means that

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{adds up to } \infty .$$

By (*), it must also be that

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n} = \infty ,$$

i.e., $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n}$ is <u>divergent</u>

Both these examples follow from :

The Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$

are series with positive terms

(i) If $\sum b_n$ is convergent

(adds up to a finite number)

and $a_n \leq b_n$ for each n,

then $\sum a_n$ is also convergent

Similarly:

(ii) If $\sum b_n$ is divergent

(adds up to ∞),

and $a_n \geq b_n$ for each n,

then $\sum a_n$ is also divergent

Ex. Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2+n+4}$ converge?

Set $b_n = \dfrac{1}{3n^2}$.

Note $a_n = \dfrac{1}{3n^2+n+4} < \dfrac{1}{3n^2} = b_n$

The p-test with $p = 2$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges}$$

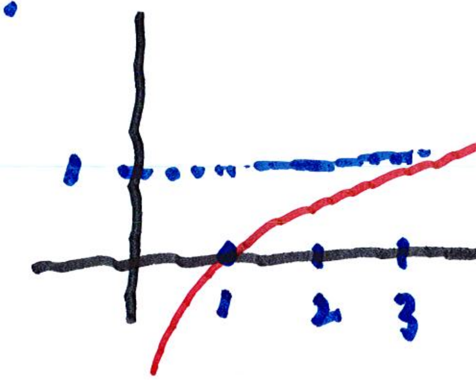
$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{3n^2} \text{ converges.}$$

$\therefore$ (i) of Comp. Test $\Rightarrow$

$$\sum_{n=1}^{\infty} \frac{1}{3n^2 + n + 4} \text{ converges}$$

Ex. What about $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}}$?

We want to say that for most n, $\ln n > 1$.



If $n \geq 3$,

then $\ln n \geq \ln 3 > \ln e = 1$

$\therefore$ If $n \geq 3$,

$$\underset{a_n}{\underset{\uparrow}{\frac{\ln n}{\sqrt{n}}}} > \underset{b_n}{\underset{\uparrow}{\frac{1}{\sqrt{n}}}}$$

Since $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

$\left(p = \frac{1}{2} \right)$

(ii) of Comp. Test

$$\Rightarrow \sum_{n=3}^{\infty} \frac{\ln n}{\sqrt{n}} \quad \text{also diverges.}$$

(We can use the Comp Test for all $n \geq N$, where N is a fixed integer)

Ex. Does $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converge or diverge?

$$\frac{\sqrt{n}}{n^2+1} \sim \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}},$$

so it "should" converge.

$$a_n = \frac{\sqrt{n}}{n^2+1} \le \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}} = b_n$$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ conv. $\left(p = \frac{3}{2}\right)$

$\therefore$ (i) $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converges.

Ex. What about $\displaystyle\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$

$$\frac{n^3}{n^4-1} \sim \frac{n^3}{n^4} = \frac{1}{n}$$

and $\sum \frac{1}{n}$ div., so series

probably diverges

Note $\frac{n^3}{n^4-1} \geq \frac{n^3}{n^4} = \frac{1}{n}$

a_n " " b_n

smaller denominator

$\Rightarrow$ bigger fraction

$\therefore$ (ii) of Comp. Test $\Rightarrow$

$\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$ diverges

Here's a problem:

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ conv.,

probably $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ does

too

Clearly $2^n - 1 < 2^n$

$$\to \frac{1}{2^n} < \frac{1}{2^n - 1} \quad (+)$$

and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ also converges

Note that ((+) is useless.)

For this problem, we have:

Limit Comparison Test.

Suppose $\sum a_n$ and $\sum b_n$

are series with positive terms

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$,

where c is finite and positive,

then either both series converge

or both diverge.

Ex. Look at $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ $\leftarrow a_n$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ $\leftarrow b_n$

Then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}}$

$\displaystyle = \lim_{n\to\infty} \frac{2^n}{2^n-1} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{2^n}} \to 1$

$= 1 > 0$

Series converges!

Ex. Does $\sum_{n=1}^{\infty} \frac{2n-1}{3n^2+2}$ conv. or div. ?

$a_n = \frac{2n-1}{3n^2+2}$. For b_n,

throw away "junk" terms. Use $b_n = \frac{2n}{3n^2} = \frac{2}{3n}$

$\therefore \sum b_n = \sum_{n=1}^{\infty} \frac{2n}{3n^2}$

$= \frac{2}{3} \sum_{n=1}^{\infty} \frac{1}{n}$ <u>diverges</u>

Show that $\lim \dfrac{\frac{2n-1}{3n^2+2}}{\frac{2}{3n}} = c > 0$

$$\lim \frac{\dfrac{2n-1}{3n^2+2}}{\dfrac{2}{3n}} = \lim \frac{3n(2n-1)}{6n^2+4}$$

$$= \frac{6n^2 - 3n}{6n^2+4} \longrightarrow 1$$

as $n \to \infty$.

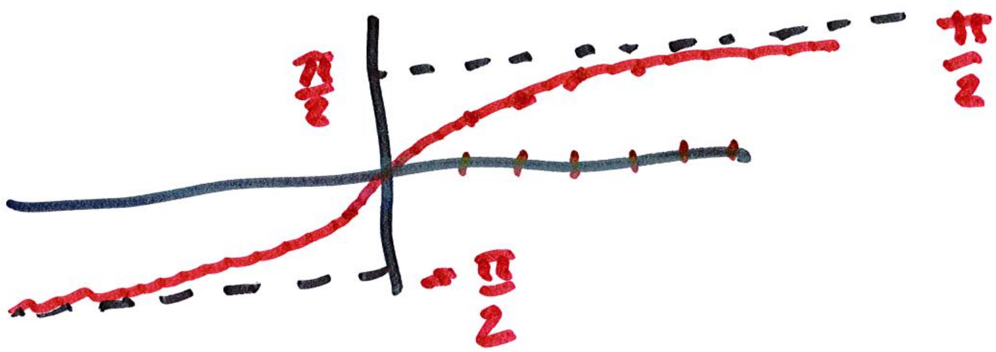
$\therefore$ Lim Comp. Test

$$\rightarrow \sum_{n=1}^{\infty} \frac{2n-1}{3n^2+2} \text{ diverges}$$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{2n-1}{3n^2+2}$ also diverges

Ex. Look at $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{\arctan n}{n^{1.2}}}_{a_n}$

As $n \to \infty$, $\arctan n \to \frac{\pi}{2}$



Use $\sum b_n = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.2}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\arctan n}{n^{1.2}}}{\frac{1}{n^{1.2}}}$$

$$= \lim_{n\to\infty} \arctan n = \frac{\pi}{2}$$

Since $\sum b_n = \sum \frac{1}{n^{1.2}}$,

the p-test implies both series converge.

Ex. Look at $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{3n^2+2}}{4n^2+2n+1}$

a_n

Look at highest order terms:

$$\frac{\sqrt{n^2}}{n^2} = \frac{n}{n^2} = \frac{1}{n} \qquad \sum \frac{1}{n}$$

b_n

$\therefore$ Probably diverges

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{\sqrt{3n^2+2}}{4n^2+2n+1}}{\dfrac{1}{n}}$$

$$= \lim_{n \to \infty} \frac{n\sqrt{3n^2+2}}{4n^2+2n+1}$$

$$= \lim_{n \to \infty} \frac{n^2\sqrt{3+2/n^2}}{n^2\left(4+\frac{2}{n}+\frac{1}{n^2}\right)}$$

$$= \lim_{n \to \infty} \frac{\sqrt{3+\frac{2}{n^2}}}{4+\frac{2}{n}+\frac{1}{n^2}} = \frac{\sqrt{3}}{4}$$

Since $\sum \frac{1}{n}$ diverges, so does $\sum a_n$