Limit Comparison Test

Let \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) be infinite series with \( a_n > 0 \) and \( b_n > 0 \). Suppose \( \lim_{n \to \infty} \frac{a_n}{b_n} = L \) where \( L > 0 \). Then either both series converge or both series diverge.
Ex. Does \[ \sum_{n=1}^{\infty} \frac{\sqrt{2n+1}}{n^2+1} \]
converge.

Set \[ b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}} \]

\[ \frac{a_n}{b_n} = \frac{\frac{\sqrt{2n+1}}{n^2+1}}{\frac{1}{n^{3/2}}} = \frac{n^{3/2} \sqrt{2n+1}}{n^2+1} \]

\[ = n^{3/2} \sqrt{n} \cdot \sqrt{2 + \frac{1}{n}} \]

\[ \frac{n^2 (1 + \frac{1}{n^2})}{1 + \frac{1}{n^2}} \]

\[ = \sqrt{2 + \frac{1}{n}} \rightarrow \frac{\sqrt{2}}{1} = \sqrt{2} \]
Since \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \) converges (by the p-test with \( p = \frac{3}{2} \)) it follows that

\[
\sum_{n=1}^{\infty} \frac{\sqrt{2n+1}}{n^2+1} \text{ converges.}
\]

\[
\sum_{n=1}^{\infty} \frac{1}{n^{2-1} \ln n} \text{ can't find } b_n.
\]

\( n \geq 3 \rightarrow \ln n > 1 \)

\[
\frac{1}{n^2 \ln n} < \frac{1}{n^2}. \ \sum \frac{1}{n^2} \text{ converges}
\]
11.6 Absolute Convergence

We say that a series \( \sum_{n=1}^{\infty} a_n \) converges absolutely if

\[ \sum_{n=1}^{\infty} |a_n| \] converges.

Ex. \[ \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^3} \] converges absolutely because

\[ \sum_{n=1}^{\infty} \left| (-1)^n \cdot \frac{1}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3} \] converges.
Ex. \[ \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n} \] does not converge absolutely because

\[ \sum_{n=1}^{\infty} \left| (-1)^n \cdot \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \] diverges.

We say a series \[ \sum_{n=1}^{\infty} a_n \] converges conditionally if \[ \sum_{n=1}^{\infty} a_n \] but \[ \sum_{n=1}^{\infty} |a_n| \] does not converge.
converge absolutely.

\[ \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \text{ converges} \]

(by the Alt. Series Test)

but does not converge absolutely. Therefore

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges conditionally.} \]
Thm. If \( \sum_{n=1}^{\infty} |a_n| \) converges absolutely, then it converges.

Pf. Note that

\[
0 \leq a_{n+1} + 1|a_n| \leq 2|a_n|
\]

Thus, by the Comparison Test, \( \sum_{n=1}^{\infty} a_{n+1} + 1|a_n| \) converges.
Then
\[ \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + l a_n) - \sum_{n=1}^{\infty} l a_n, \]

because the difference of two convergent series is convergent.

Given a series \( \sum_{n=1}^{\infty} a_n \),
there are 3 possibilities:

1. $\sum_{n=1}^{\infty} a_n$ does not converge

2. $\sum_{n=1}^{\infty} |a_n|$ converges conditionally

3. $\sum_{n=1}^{\infty} a_n$ converges absolutely
Ex. Show that \[ \sum_{n=1}^{\infty} \frac{\cos n}{n^2} \] is convergent.

Note that this series is NOT alternating, so we can't apply the Alt. Series Test.

We can show the series is absolutely convergent.
\[
\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}.
\]

This series is convergent by the p-test with \( p = 2 \).

The Comparison Test implies that \( \sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| \) is convergent.

Hence, the series \( \sum_{n=1}^{\infty} \frac{\cos n}{n^2} \)
is convergent (by the above Theorem).

The following test can be used to show that a series is absolutely convergent.
The Ratio Test.

(i) If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \).

Then \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent.

(ii) If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1 \), or \( \infty \),

Then \( \sum_{n=1}^{\infty} a_n \) diverges.

(iii) If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \),
Then the Ratio Test is inconclusive; it might be that \[ \sum_{n=1}^{\infty} a_n \]
converges or not.

**Ex.** Show that \[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n+2)}{2^n} \]

The Ratio Test works well if \( a_n \) contains geometric factors like \( 2^n \) or factorials like \( n! \).
\[ \left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1) + 2}{2^{n+1}} \]
\[ \frac{1}{(n+2) 2^{n+1}} \]
\[ = \frac{(n+3) \cdot 2^n}{(n+2) 2^{n+1}} \]
\[ = \frac{1}{2} \cdot \frac{n(1 + \frac{3}{n})}{n(1 + \frac{2}{n})} \cdot \frac{1}{2} \]
\[ = \frac{1 + \frac{3}{n}}{(1 + \frac{2}{n})} \cdot \frac{1}{2} \rightarrow \frac{1}{2} \quad \text{as} \ n \rightarrow \infty \]
\[ L = \frac{1}{2} < 1 \text{, so by \((i)\), the series converges absolutely.} \]

Ex. Does \( \sum_{n=1}^{\infty} \frac{n!}{3^n} \) converge?

\[ \left( \text{We don't need absolute value signs} \right) \]

\[ \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{3^{n+1}} \leq \frac{n!}{3^n} \]

\[ = \frac{(n+1)! \cdot 3^n}{n! \cdot 3^{n+1}} \]
\[(n+1)! = (n+1) \cdot n \cdot (n-1) \cdots 2 \cdot 1\]

\[\frac{1}{n!} \cdot \frac{n \cdot (n-1) \cdots 2 \cdot 1}{n(n-1) \cdots 2 \cdot 1} = (n+1)\]

\[\frac{(n+1)!}{n! \cdot 3^{n+1}} = \frac{(n+1)}{3} \rightarrow \infty\]

\[\sum_{n=1}^{\infty} \frac{n!}{3^n} \text{ diverges by (ii).}\]
Ex. Does \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \) converge.

Clearly the series converges by the Alt. Series Test, but let's apply the Ratio Test.

\[
\left| \frac{\frac{(-1)^{n+1}}{\sqrt{n+1}}}{\frac{(-1)^n}{\sqrt{n}}} \right| = \frac{\sqrt{n}}{\sqrt{n+1}} = \sqrt{\frac{n}{n+1}}
\]

\[= \sqrt{\frac{n}{n(1+\frac{1}{n})}} = \sqrt{1+\frac{1}{n}} = 1. \quad (\text{iii})\]
Now consider \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \), which diverges by the p-test with \( p = \frac{1}{2} \).

If we apply the Ratio Test,

\[
\frac{a_{n+1}}{a_n} = \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \sqrt{\frac{n}{n+1}} \rightarrow 1.
\]

\( \therefore L = 1 \) doesn't help.
Why does the Ratio Test work?

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L, \text{ where } L < 1. \]

\[ \therefore \left| \frac{a_{n+1}}{a_n} \right| \leq L, \text{ all } n \geq N \]

\[ \Rightarrow \left| a_{N+1} \right| \leq \left| a_N \right| \]

\[ \left| a_{N+1} \right| \leq \left| a_N \right| L \]

\[ \left| a_{N+2} \right| \leq \left| a_N \right| L^2 \]

\[ \therefore \left| a_{N+k} \right| \leq \left| a_N \right| L^k \]
\[
\sum_{k=0}^{\infty} \left| a_{N+k} \right| \leq 1 |a_N| \sum_{k=0}^{\infty} \ell^k
\]

Hence this series is convergent.

\[\sum_{k=0}^{\infty} a_{N+k} \text{ is convergent.}\]
\[ \left| \frac{a_{n+1}}{a_n} \right| \geq 1, \quad \forall n \in \mathbb{N} \]

\[ \Rightarrow \left| a_{n+1} \right| \geq \left| a_n \right|, \quad n \in \mathbb{N}. \]

\[ \therefore \left| a_n \right| \to \infty \]

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Ex. Does \[ \sum_{n=1}^{\infty} \frac{n! \cdot n!}{(2n)!} \] converge?

\[ a_{n+1} = \frac{(n+1)! \cdot (n+1)!}{(2(n+1))!} \]

\[ \frac{n! \cdot n!}{(2n)!} \]
\[
\frac{(n+1)!}{(2n+2)!} \cdot \frac{n!}{n!} = \frac{(n+1)!^2}{(2n+2)!}
\]

\[
\Rightarrow \frac{(n+1)^2}{(2n+2)(2n+1)} \rightarrow \frac{1}{4} < 1.
\]

As \( n \to \infty \)

\( \therefore \) Series converges