Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

be infinite series with $a_n > 0$

and $b_n > 0$. Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = L$

where $L > 0$. Then either

both series converge

or both series diverge.

Ex. Does $\sum_{n=1}^{\infty} \frac{\sqrt{2n+1}}{n^2+1}$ converge.

Set $b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

$$\frac{a_n}{b_n} = \frac{\frac{\sqrt{2n+1}}{n^2+1}}{\frac{1}{n^{3/2}}} = \frac{n^{3/2}\sqrt{2n+1}}{n^2+1}$$

$$= \frac{n^{3/2}\sqrt{n}\cdot\sqrt{2+\frac{1}{n}}}{n^2\left(1+\frac{1}{n^2}\right)}$$

$$= \frac{\sqrt{2+\frac{1}{n}}}{1+\frac{1}{n^2}} \to \frac{\sqrt{2}}{1} = \sqrt{2}$$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

converges (by the p-test

with $p = \frac{3}{2}$) it follows that

$\sum_{n=1}^{\infty} \frac{\sqrt{2n+1}}{n^2+1}$ converges.

$\sum_{n=1}^{\infty} \frac{1}{n^2 \ln n}$ can't find b_n.

$n \geq 3 \rightarrow \ln n > 1$

$\frac{1}{n^2 \ln n} < \frac{1}{n^2}$. $\sum \frac{1}{n^2}$ conv

11.6 Absolute Convergence

We say that a series $\sum_{n=1}^{\infty} a_n$

converges absolutely if

$\sum_{n=1}^{\infty} |a_n|$ converges.

Ex. $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^3}$ converges

absolutely because

$\sum_{n=1}^{\infty} \left| (-1)^n \cdot \frac{1}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

Ex. $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ does <u>not</u>

converge absulutely because

$$\sum_{n=1}^{\infty} \left| (-1)^n \cdot \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

We say a series $\sum_{n=1}^{\infty} a_n$

converges conditionally if

$$\underbrace{\sum_{n=1}^{\infty} a_n}_{\text{converges}} \text{ but } \sum_{n=1}^{\infty} |a_n| \text{ does } \underline{\text{not}}$$

converge absolutely.

Ex. $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ converges

(by the Alt. Series Test)

but does not converge

absolutely. Therefore

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally.

Thm.

If $\sum_{n=1}^{\infty} a_n$ converges absolutely,

then it converges.

Pf. Note that

$$0 \le a_n + |a_n| \le 2|a_n|$$

Thus, by the Comparison Test,

$\sum_{n=1}^{\infty} a_n + |a_n|$ converges.

Then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n| ,$$

because the difference of two convergent series is convergent.

Given a series $\sum_{n=1}^{\infty} a_n$, there are 3 possibilities:

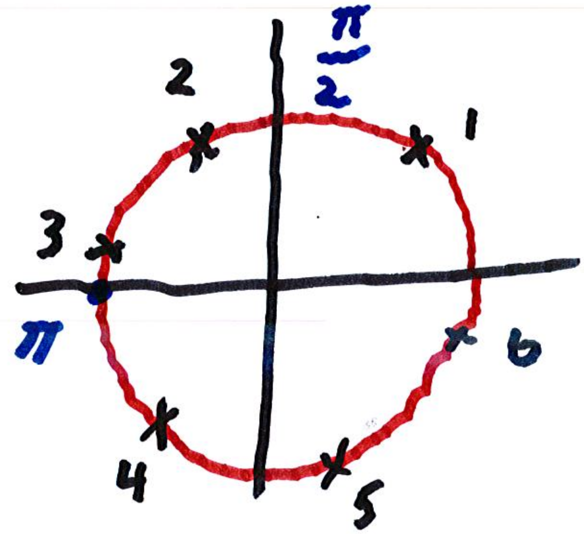
1. $\sum_{n=1}^{\infty} a_n$ does not converge

2. $\sum_{n=1}^{\infty} a_n$ converges conditionally

3. $\sum_{n=1}^{\infty} a_n$ converges absolutely

Ex. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent.

Note that this series is NOT alternating, so we can't apply the Alt. Series Test



We can show the series is absolutely convergent.

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This series is convergent by the p-test with $p = 2$.

The Comparison Test implies that $\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right|$ is convergent

Hence, the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

is convergent [by the above Theorem].

The following test can be used to show that a series is absolutely convergent.

The Ratio Test.

(i) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1.$

Then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or ∞,

Then $\sum_{n=1}^{\infty} a_n$ diverges

(iii) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, Then the Ratio Test is

inconclusive; it might be

that $\sum_{n=1}^{\infty} a_n$ converges or not.

Ex. Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n+2)}{2^n}$.

(The Ratio Test works well if a_n contains geometric factors like 2^n or factorials like $n!$)

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\dfrac{(n+1)+2}{2^{n+1}}}{\dfrac{(n+2)}{2^n}}$$

$$= \frac{(n+3)\cdot 2^n}{(n+2)\, 2^{n+1}}$$

$$= \frac{(n+3)}{(n+2)} \cdot \frac{1}{2} = \frac{n\left(1+\frac{3}{n}\right)}{n\left(1+\frac{2}{n}\right)} \cdot \frac{1}{2}$$

$$= \frac{\left(1+\frac{3}{n}\right)}{\left(1+\frac{2}{n}\right)} \cdot \frac{1}{2} \longrightarrow \frac{1}{2} \quad \text{as } n \to \infty$$

$L = \frac{1}{2} < 1$, so by (i),

the series converges absolutely.

Ex. Does $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^n}$ converge?

(We don't need absolute value signs)

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)!}{3^{n+1}}}{\dfrac{n!}{3^n}} = \frac{(n+1)! \cdot 3^n}{n! \cdot 3^{n+1}}$$

$$\frac{(n+1)!}{n!} = \frac{(n+1)\cdot n\cdot (n-1)\ldots 2\cdot 1}{n(n-1)\ldots 2\cdot 1}$$

$$= (n+1)$$

$$\therefore \quad \frac{(n+1)!\cdot 3^n}{n!\ 3^{n+1}} = \frac{(n+1)}{3} \to \infty$$

$\therefore \quad \sum_{n=1}^{\infty} \frac{n!}{3^n}$ diverges by (ii).

Ex. Does $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converge.

Clearly the series converges by the Alt. Series Test, but let's apply the Ratio Test.

$$\left| \frac{\frac{(-1)^{n+1}}{\sqrt{n+1}}}{\frac{(-1)^n}{\sqrt{n}}} \right| = \frac{\sqrt{n}}{\sqrt{n+1}} = \sqrt{\frac{n}{n+1}}$$

$$= \sqrt{\frac{n}{n(1+\frac{1}{n})}} = \frac{1}{\sqrt{1+\frac{1}{n}}} \to 1. \quad \text{(iii)}$$

Now consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$,

which diverges by the p-test

with $p = \frac{1}{2}$.

If we apply the Ratio Test,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \sqrt{\frac{n}{n+1}} \to 1.$$

$\therefore L = 1$ doesn't help.

Why does the Ratio Test work?

where $\ell < 1$.

$$\therefore \left| \frac{a_{n+1}}{a_n} \right| \leq \ell \ , \quad \text{all } n \geq N$$

$|a_{n+1}| \leq |a_n| \ell$ if $n \geq N$

$$\Rightarrow \quad |a_N| \leq |a_N|$$

$$|a_{N+1}| \leq |a_N| \ell$$

$$|a_{N+2}| \leq |a_N| \ell^2$$

$$\vdots$$

$$|a_{N+k}| \leq |a_N| \ell^k$$

$$\vdots$$

$$\therefore \sum_{k=0}^{\infty} |a_{N+k}| \leq |a_N| \underbrace{\sum_{k=0}^{\infty} \ell^k}_{\text{convergent.}}$$

Hence this series is absolutely convergent

$$\therefore \sum_{k=0}^{\infty} a_{N+k} \text{ is convergent.}$$

$\ell > 1.$

$$\therefore \left| \frac{a_{n+1}}{a_n} \right| \geq \ell, \quad \text{all } n \geq N$$

$$\rightarrow |a_{n+1}| \geq |a_n| \ell, \quad n \geq N.$$

$$\therefore |a_n| \rightarrow \infty$$

Ex. Does $\displaystyle\sum_{n=1}^{\infty} \frac{n!\, n!}{(2n)!}$ converge?

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{(n+1)!\,(n+1)!}{(2(n+1))!}}{\dfrac{n!\, n!}{(2n)!}}$$

$$= \frac{\dfrac{(n+1)!\,(n+1)!}{(2n+2)!}}{\dfrac{n!\cdot n!}{(2n)!}}$$

$$= \frac{(n+1)!\,(n+1)!\;(2n)!}{n!\cdot n!\;(2n+2)!}$$

$$= \frac{(n+1)^2}{(2n+1)(2n+2)} \to \frac{1}{4} < 1.$$

as $n \to \infty$

$\therefore$ Series converges