11.8 Power Series

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where $x$ is a variable and the $c_n$'s are the coefficients of the series.

A power series may converge for some values of $x$ and diverge for others.
If we set \( c_n = 1 \) for all \( n \), we obtain

\[
\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots + x^n + \ldots,
\]

which converges to \( \frac{1}{1-x} \) if \( |x| < 1 \) and diverges if \( |x| \geq 1 \).

More generally, the series

\[
\sum_{n=0}^{\infty} c_n (x-a)^n
\]

\( = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots + c_n (x-a)^n + \ldots \)
is called a series in \((x-a)\)
or a power series centered
at \(a\) or about \(a\). The
series in (1) obviously
converges when \(x = a\).

We often use the
**Ratio Test** to determine
for which \(x\) the series
converges.
Ex. For which $x$ does

$$\sum_{n=0}^{\infty} n! \cdot x^n \text{ converge?}$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)! \cdot x^{n+1}}{n! \cdot x^n} \right|$$

$$= (n+1) |x| \to \infty \quad \text{if} \quad x \neq 0.$$ 

Therefore

$$\sum_{n=0}^{\infty} n! \cdot x^n \text{ only converges if} \quad x = 0.$$
Ex. For which \( x \) does

\[
\sum_{n=0}^{\infty} \frac{x^n}{(n+2) 2^n}
\]

converge?

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+3) 2^{n+1}} \right| \frac{x^n}{(n+2) 2^n}
\]

\[
= \left| \frac{x}{2} \right| \frac{(n+2)}{(n+3)}
\]

\[
\rightarrow \left| \frac{x}{2} \right| \text{ as } n \rightarrow \infty \quad (\text{if } x \neq 0).
\]
\[ \sum_{n=1}^{\infty} \frac{x^n}{(n+2)2^n} \text{ converges if } \frac{|x|}{2} < 1, \text{ i.e. if } |x| < 2. \]

Also the series diverges if \( \frac{|x|}{2} > 1 \), i.e., if \( |x| > 2 \)

If \( x = 2 \), the series is \[ \sum_{n=1}^{\infty} \frac{1}{n+2} \]
which diverges
If \( x = -2 \), the series is

\[
\sum_{n=1}^{\infty} \frac{(-2)^n}{(n+2)2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+2)},
\]

which converges by the Alternating Series Test.

Thus the series converges if \( -2 \leq x < 2 \)

and diverges elsewhere.
Ex. For which $x$ does

$$\sum_{n=1}^{\infty} \frac{2^n (x-2)^n}{(n+1)^2}$$

converge?

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{2^{n+1} (x-2)^{n+1}}{(n+2)^2} - \frac{2^n (x-2)^n}{(n+1)^2} \right|$$

$$= \left| \frac{2^{n+1} (x-2)^{n+1}}{(n+2)^2} - \frac{(n+1)^2}{2^n (x-2)^n} \right|$$
\[ = 2 |x-2| \cdot \frac{(n+1)^2}{(n+2)^2} \rightarrow 2|x-2| \quad \text{as } n \rightarrow \infty \]

\[ \therefore \text{If } 2|x-2| < 1, \text{ i.e. } |x-2| < \frac{1}{2} \]

then the series converges.

Also, if \( 2|x-2| > 1, \text{ i.e., if } |x-2| > \frac{1}{2} \), then the series diverges.
If \( x = \frac{5}{2} \), the series is

\[
\sum_{n=1}^{\infty} \frac{2^n \cdot \left( \frac{1}{2} \right)^n}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}
\]

converges.

If \( x = \frac{3}{2} \), then \((x-2)^n = \left( -\frac{1}{2} \right)^n\).

so the series is

\[
\sum_{n=1}^{\infty} \frac{2^n \left( -\frac{1}{2} \right)^n}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^2}
\]

which also converges.
Thus the series when

\[ \frac{3}{2} \leq x \leq \frac{5}{2} \]

Ex. Find all \( x \) where

\[ \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \quad (a! = 1) \]

converges.

\[ \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{1}{x^n} \cdot \frac{1}{n!} \right| \]
\begin{align*}
\left| \frac{x^{n+1}}{(n+1)!} \right| & \cdot \left| \frac{n!}{x^n} \right| \\
= \frac{|x|}{n+1} & \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{align*}

\[ \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ converges for all } x. \]

Given any power series, it always converges in a symmetric open interval.
Thm. For a power series

\[ \sum_{n=0}^{\infty} c_n (x-a)^n, \] there are only 3 possibilities:

(i) The series converges only when \( x = a \).

(ii) The series converges for all \( x \).

(iii) There is a number \( R > 0 \) so that the series converges when \( |x-a| < R \) and diverges when \( |x-a| > R \).
The number $R$ in (iii) is called the Radius of Convergence. In case (i), we set $R = 0$, and in case (iii), we set $R = \infty$.

The above theorem says nothing about the cases when

$$x = a - R \quad \text{or} \quad x = a + R.$$
So far, we have

<table>
<thead>
<tr>
<th>Series</th>
<th>Radius of Convergence</th>
<th>Interval of Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \sum_{n=0}^{\infty} x^n ]</td>
<td>1</td>
<td>((-1, 1])</td>
</tr>
<tr>
<td>[ \sum_{n=0}^{\infty} n! x^n ]</td>
<td>0</td>
<td>({0})</td>
</tr>
<tr>
<td>[ \sum_{n=1}^{\infty} \frac{x^n}{(n+2)2^n} ]</td>
<td>2</td>
<td>([-2, 2)]</td>
</tr>
<tr>
<td>[ \sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{(n+1)^2} ]</td>
<td>(\frac{1}{2})</td>
<td>(\left[\frac{3}{2}, \frac{5}{2}\right])</td>
</tr>
<tr>
<td>[ \sum_{n=0}^{\infty} \frac{x^n}{n!} ]</td>
<td>(\infty)</td>
<td>((-\infty, \infty))</td>
</tr>
</tbody>
</table>
The advantage of power series is that we can greatly enlarge the set of functions.

Ex. Find the set of convergence of the Bessel function

\[ J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2} \]

\[ \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{2n+2}}{2^{2n+2} (n+1)!^2} \cdot \frac{2^{2n} (n!)^2}{x^{2n}} \right| \]

\[ x \neq 0 \]
\[
= \frac{1 \times 1}{2^2} \cdot \frac{1}{(n+1)^2} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Hence the power series converges for all \( x \), and so \( R = \infty \).

Find the set of convergence of
\[
\sum_{n=0}^{\infty} \frac{(-2)^n x^n}{\sqrt{n+4}}
\]
\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{2^{n+1} 1x1^{n+1}}{\sqrt{n+5}} \cdot \frac{\sqrt{n+4}}{2^n 1x1^n}
\]
\[
= 2 1x1 \sqrt{\frac{n+4}{n+5}} \to 2 1x1.
\]

:. Series converges if \(2 1x1 < 1\)

\[
\to 1x1 < \frac{1}{2}.
\]

and Series diverges if \(2 1x1 > 1\)

\[
\to 1x1 > \frac{1}{2}.
\]
Look at endpoints

$x = \frac{1}{2}$ and $x = -\frac{1}{2}$.

\[
x = \frac{1}{2} \rightarrow \sum_{n=0}^{\infty} \frac{(-2)^n}{\sqrt{n+4}} \frac{1}{2^n}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}
\]

which converges by the Alt. Series Test
When \( x = -\frac{1}{2} \),

\[
\sum_{n=0}^{\infty} \frac{(-2)^n \cdot (-\frac{1}{2})^n}{\sqrt{n+4}}
\]

\( = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+4}} \), which diverges

by the Limit Comp. Test.

\( R = \frac{1}{2} \), Int. of Conv. is \((-\frac{1}{2}, \frac{1}{2}]\)
Why is the Interval of Convergence Symmetric?

Suppose \( \sum_{n=0}^{\infty} c_n r^n \) converges.

Then the sequence \( |c_n r^n| \) is bounded.

\[ \therefore \text{There is a constant } M \text{ so } |c_n r^n| \leq M. \]
Now let $x$ be any number with $|x| < n$. Then

\[
\sum_{n=0}^{\infty} |c_n x^n| \leq \sum_{n=0}^{\infty} |c_n n^n| 1 \cdot \frac{|x|^n}{n^n} \\
\leq \sum_{n=0}^{\infty} M \cdot \frac{|x|^n}{n^n} \\
= M \cdot \frac{1}{1-\frac{|x|}{n}}.
\]

(since the series is a geometric series.)
Thus the series \( \sum_{n=0}^{\infty} c_n x^n \) converges (by the Abs. Conv. Test).

If we can find \( R \) to be arbitrarily large, then \( \sum_{n=0}^{\infty} c_n x^n \) conv. for all \( x \).
Otherwise, we can find  $\mathbf{r}$ as close as we want to $\mathbf{R}$.