11.9 Representation of Functions as Power Series

The most basic power series is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n,$$

which converges when $|x| < 1$

If we replace x by $-x$, then

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \dots$$

$$= 1 - x + x^2 - x^3 + \dots$$

which converges if $|-x| < 1$.

$$\therefore \frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots = \sum_{n=0}^{\infty} (-1)^n x^n$$

which also converges if $|x| < 1$.

By substitution, we get

$$\frac{1}{2-3x} = \frac{1}{2\left(1 - \frac{3x}{2}\right)} = \frac{1}{2(1-u)}, \quad |u| < 1$$

$$u = \frac{3x}{2}$$

$$= \frac{1}{2}\left(1 + \frac{3x}{2} + \left(\frac{3x}{2}\right)^2 + \dots\right)$$

$$\frac{1}{2-3x} = \frac{1}{2} + \frac{3x}{2^2} + \frac{3^2 x^2}{2^3} + \dots$$

$$\therefore \frac{1}{2-3x} = \sum_{n=0}^{\infty} \frac{3^n x^n}{2^{n+1}} .$$

This series converges if

$\left| \frac{3x}{2} \right| < 1$, i.e., $\underline{|x| < \cancel{\frac{3}{2}}\ \frac{2}{3}}$

Ex. Find the power series

of $\frac{x^2}{3+4x}$.

First $\frac{1}{3+4x} = \frac{1}{3\left(1+\frac{4x}{3}\right)}$

$$= \frac{1}{3}\left(1 - \frac{4x}{3} + \left(\frac{4x}{3}\right)^2 - \left(\frac{4x}{3}\right)^3 + \cdots\right)$$

$$= \frac{1}{3} - \frac{4x}{3^2} + \frac{4^2x^2}{3^3} - \frac{4^3x^3}{3^4} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{4^n x^n}{3^{n+1}}$$

If we multiply by x^2, we get

$$\frac{x^2}{3+4x} = \frac{x^2}{3} - \frac{4x^3}{3^2} + \frac{4^2x^4}{3^3} - \cdots$$

or

$$\frac{x^2}{3+4x} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{n+2}}{3^{n+1}} .$$

This series converges

if $\left| \frac{4x}{3} \right| < 1$ or $|x| < \frac{3}{4}$

(Look where the substitution

was made.)

It turns out we can differentiate and integrate power series.

Thm. If the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$, then the function

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cancel{c_3(x-a)^3} + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$

$$= \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$ and

(ii) $$\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \dots$$

or

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}.$$

Also the radius of the power series in (i) and (ii) equals R.

In other words, the power series of f' and $\int f$ can be obtained by differentiating and integrating each term.

Ex. Calculate the power series of $\frac{1}{(1+x)^2}$ and $\frac{1}{(1+x)^3}$.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

$$-\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 + \dots$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \dots$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} n x^{n-1}$$

Now differentiate again:

$$\frac{-2}{(1+x)^3} = -2 + 3\cdot 2x - 4\cdot 3x^2 + 5\cdot 4x^3 - \cdots$$

$$= \sum_{n=2}^{\infty} (-1)^{n-1} \frac{n(n-1)}{2} x^{n-2}$$

Now divide by (-2).

$$\frac{1}{(1+x)^3} = 1 - \frac{3\cdot 2}{2}x + \frac{4\cdot 3}{2}x^2 - \frac{5\cdot 4}{2}x^3 + \cdots = \sum_{n=2}^{\infty} (-1)^n \frac{n(n-1)}{2} x^{n-2}$$

Ex. Find the power series of $\ln(1+x)$. First

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

if $|x| < 1$.

Now integrate both sides:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + C$$

If $x = 0$

$$\ln(1+0) = 0 - 0 + 0 \cdots + C$$

$$\begin{array}{c} \| \\ 0 \end{array} \qquad \therefore C = 0.$$

$$\therefore \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n} x^{n+1}}{n+1} \quad \{\text{if } |x| < 1\}$$

Ex. Find the power series of

$$f(x) = \frac{x^2}{1-x^2}$$

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots$$

$|u| < 1$.

Set $u = x^2$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \dots$$

if $|x^2| < 1$

i.e $|x| < 1$

Multiply by x^2

$$\frac{x^2}{1-x^2} = x^2 + x^4 + x^6 + \cdots$$

if $|x| < 1$.

Ex. Find the power series of $\tan^{-1} x$. Note that

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

if $|x| < 1$

Now integrate:

$$\tan^{-1} x = \int \frac{dx}{1+x^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + C$$

Again, if $x = 0$,

$$0 = C + 0 - 0 + 0 \ldots$$

$\therefore \; C = 0$

$$\rightarrow \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

if $|x| < 1$.

Ex. Find the power series of

$$\frac{\tan^{-1} x}{x}$$

Just divide by x:

$$\frac{\tan^{-1}x}{x} = 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots$$

if $|x| < 1$.

Note that $\frac{\tan^{-1}x}{x}$ can be differentiated as many times as we like, inside $(-1, 1)$.

Ex. Find the power series of

$$\frac{1}{1+x^5}.$$

$$\frac{1}{1+u} = 1 - u + u^2 - u^3 + \dots$$

in $|u| < 1.$

Set $u = x^5$ and find the

integral of $\frac{1}{1+x^5}$ =

$$1 - x^5 + x^{10} - x^{15} + \dots$$

$$\int \frac{dx}{1+x^5} = x - \frac{x^6}{6} + \frac{x^{11}}{11} - \frac{x^{16}}{16} + \cdots + C$$

$$\int_0^t \frac{dx}{1+x^5} = C + t - \frac{t^6}{6} + \frac{t^{11}}{11} - \frac{t^{16}}{16} + \cdots$$

$C = 0$

Now suppose that $0 \le t \le \frac{1}{2}$

We can use the Alternating Series Estimate.

$$\left| \int_0^t \frac{dx}{1+x^5} - \left(t - \frac{t^6}{6} + \frac{t^{11}}{11} \right) \right| < \frac{t^{16}}{16}$$

If $0 \le t \le \frac{1}{2}$, then

$$\frac{\left(\frac{1}{2}\right)^{16}}{2^4} = \frac{1}{2^{20}} < 10^{-6}.$$

$\therefore$ For all t with $0 \le t \le \frac{1}{2}$,

$\left(t - \frac{t^6}{6} + \frac{t^{11}}{11}\right)$ approximates

$\int_0^t \frac{dx}{1+x^5}$ to within 10^{-6}.

Ex. Use partial fractions to find the power series of $f(x) = \dfrac{x+2}{2x^2 - x - 1}$

$$\frac{x+2}{2x^2 - x - 1} = \frac{x+2}{(2x+1)(x-1)}$$

$$= \frac{A}{2x+1} + \frac{B}{x-1}$$

$\rightarrow A(x-1) + B(2x+1) = x+2$

$\rightarrow 2B + A = 1$

$B - A = 2$

$\therefore A = -1$ and $B = 1$

$$\rightarrow f(x) = -\sum_{n=0}^{\infty} x^n - \sum_{n=0}^{\infty} (-2x)^n$$

$$f(x) = \sum_{n=0}^{\infty} \left(-1 + (-1)^{n+1} \cdot 2^n\right) x^n$$

The radius of convergence

is 1 for first series

and $\frac{1}{2}$ for the second.

$\therefore$ Radius $= \frac{1}{2}$ for the

whole series.

Ex Find power series for

$$\frac{1}{(2-3x)^2}$$

$$= \frac{1}{4\left(1-\frac{3x}{2}\right)^2}$$

$$\frac{1}{1-u} = 1 + u + u^2 + \cdots$$

$$\frac{1}{(1-u)^2} = 1 + 2u + 3u^2 + \cdots$$

$$\frac{1}{4\left(1-\frac{3x}{2}\right)} = \frac{1}{4}\cdot\left(1 + \frac{3x}{2} + \left(\frac{3x}{2}\right)^2 + \cdots\right)$$