11.10 Taylor and Maclaurin Series

Given a function $f(x)$ defined near $x = a$, we want to find coefficients c_n, $n = 0, 1, 2, \ldots$, such that $f(x)$ can be written as a power series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$$

$$= \sum_{n=0}^{\infty} c_n (x-a)^n \text{ when } |x-a| < R$$

We write

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

If $x = a$, then

$f(a) = c_0$. If we differentiate:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

If $x = a$, then

$f'(a) = c_1$. If we differentiate:

$$f''(x) = 2c_2 + 3 \cdot 2 \cdot c_3(x-a) + \cdots$$

If $x = a$, then

$f''(a) = 2c_2$ If we differentiate

$$f'''(x) = 3\cdot 2\cdot c_3 + 4\cdot 3\cdot 2 c_4 (x-a) + \dots$$

If $x = a$, then

$f'''(a) = 3\cdot 2\cdot c_3$ If we differentiate:

$$f^{(4)}(x) = 4\cdot 3\cdot 2\cdot c_4 + \cdots$$

If $x = a$, then

$$f^{(4)}(a) = 4\cdot 3\cdot 2\cdot c_4$$

$$\vdots$$

In general,

$$f^{(n)}(a) = n!\, c_n$$

$$\text{or } c_n = \frac{f^{(n)}(a)}{n!}$$

Thm. If f has a power series representation

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then $$c_n = \frac{f^{(n)}(a)}{n!}$$

Hence $$f(x) \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

or $$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + $$

This is called the Taylor series of f at a.

If $a=0$,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

its called the Maclaurin series at a.

Ex. Find the Maclaurin series of $f(x) = e^x$.

$f(0) = e^0 = 1$

$f'(x) = e^x \rightarrow f'(0) = e^0 = 1$

$\vdots$

$f^{(n)}(x) = e^x \rightarrow f^{(n)}(0) = e^0 = 1.$

$\therefore$ Taylor Series of e^x is

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{|x|^{n+1}}{(n+1)!}}{\frac{|x|^n}{n!}} = \frac{|x|}{n+1} \rightarrow 0 \quad \underline{\underline{\text{as } n \rightarrow \infty}}$$

$\therefore$ Series converges for all x.

BIG QUESTION IS:

Does the series converge to $f(x)$?

If so, this would mean

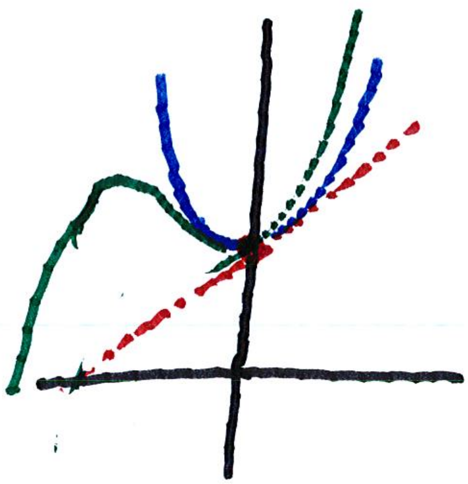
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

We set $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k$.

For $f(x) = e^x$

$T_1(x) = 1 + x \qquad T_2(x) = 1 + x + \frac{x^2}{2!}$

$$T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$



Set $R_n(x) = f(x) - T_n(x)$

If $\lim\limits_{n\to\infty} T_n(x) = f(x)$

this would mean $\lim\limits_{n\to\infty} f(x) - T_n(x) = 0$

i.e., $\lim\limits_{n\to\infty} R_n(x) = 0.$

we need an estimate of $|R_n(x)|$

that shows that $\lim\limits_{n\to\infty} |R_n(x)| = 0.$

Taylor's Inequality Does the Job!

Suppose M_{n+1} satisfies

$|f^{(n+1)}(x)| \leq M_{n+1}$, for all x with $|x-a| \leq d$.

Then the remainder $R_n(x)$ of the Taylor Series $\sum_{k=0}^{\infty} f^{(k)}(a)(x-a)^k$

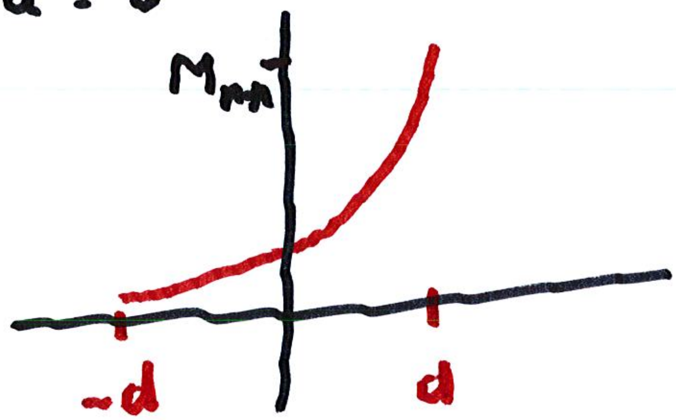
satisfies

$$|R_n(x)| \leq \frac{M_{n+1}}{(n+1)!}|x-a|^{n+1}$$

for $|x-a| \leq d$.

Show e^x equals its Maclaurin series.

Set $a = 0$



$|f^{(n+1)}(x)| = e^x \le e^d$ $\left(\begin{array}{c}\text{since } e^x \\ \text{is increasing}\end{array}\right)$

$\therefore$ Set $M_{n+1} = e^d$

Tay. Ineq. $\rightarrow$ If $|x| \le d$, then

$$|R_n(x)| \le \frac{M_{n+1}}{(n+1)!}|x|^{n+1}$$

$$= \frac{e^d |x|^{n+1}}{(n+1)!} \quad \text{as } n \to \infty$$

$$\left(\begin{array}{l} \text{To show } \lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0, \\ \\ \text{use the Ratio Test.} \end{array} \right)$$

$$\underbrace{-\frac{e^d |x|^{n+1}}{(n+1)!}}_{\downarrow \, 0} \le R_n(x) \le \underbrace{\frac{e^d |x|^{n+1}}{(n+1)!}}_{\downarrow \, 0}$$

Then use the Squeeze Thm.

$$\therefore \; e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ex. Find the Taylor Series

of $f(x) = x^4 - 2x^3 - 1$, at $a = 1$.

$f(1) = 1 - 2 - 1$

$= -2$

$f'(x) = 4x^3 - 6x^2$

$f'(1) = 4 - 6$

$= -2$

$f''(x) = 12x^2 - 12x$

$f''(1) = 0$

$f'''(x) = 24x - 12$

$f'''(1) = 12$

$f^{(4)}(x) = 24$

$f^{(4)}(1) = 24$

$f^{(5)}(x) = 0 \leftarrow M_5 = 0$

$$\therefore \quad c_0 = f(1) = -2$$

$$c_1 = \frac{f'(1)}{1!} = -2$$

$$c_2 = \frac{f''(1)}{2!} = 0$$

$$c_3 = \frac{f'''(1)}{3!} = \frac{12}{6} = 2$$

$$c_4 = \frac{f^{(4)}(1)}{4!} = \frac{24}{4!} = 1$$

$$\therefore f(x) = -2 - 2(x-1) + 2(x-1)^3 + 1\cdot(x-1)^4$$

Note $R_4(x) = 0$ since $M_5 = 0$

$f(x) = \sin x$ $\quad f(0) = 0$

$c_0 = 0$

$f'(x) = \cos x$ $\quad f'(0) = 1$

$c_1 = \frac{1}{1!} = 1$

$f''(x) = -\sin x$ $\quad f''(0) = 0$

$c_2 = 0$

$f'''(x) = -\cos x$ $\quad f'''(0) = -1$

$c_3 = \frac{-1}{3!}$

$f^{(4)}(x) = \sin x$ $\quad f^{(4)}(0) = 0$

$c_4 = 0$

$f^{(5)}(x) = \cos x$ $\quad f^{(5)}(0) = 1$

$c_5 = \frac{1}{5!}$

etc.

$\therefore$ Maclaurin Series of $\sin x$

is

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

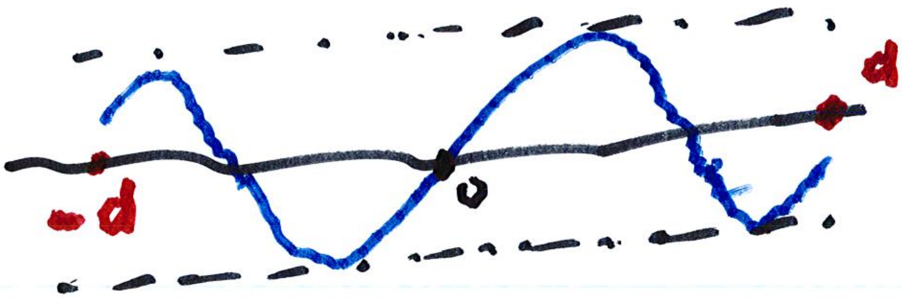
Why does the above series

converge to $\sin x$?

If $f(x) = \sin x$, then

$$f^{(n+1)}(x) = \pm \sin x \quad \text{or} \quad \pm \cos x.$$

Set $M_{n+1} = 1$, and d = any number ≥ 0.



Then $|f^{(n+1)}(x)| \le 1$,

all $|x| \le d$.

$$\therefore |R_n(x)| \le \frac{1 \cdot |x|^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=0} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$$

Similarly, it follows that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

all x.

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{all } x.$$

Ex. Find the Taylor series

of $f(x) = \sqrt{x}$ about $\underline{x = 4}$

(Find first 4 terms.