10.3 Polar Coordinates

If P is a point in the plane, we define the polar coordinates of P by (r, θ), where

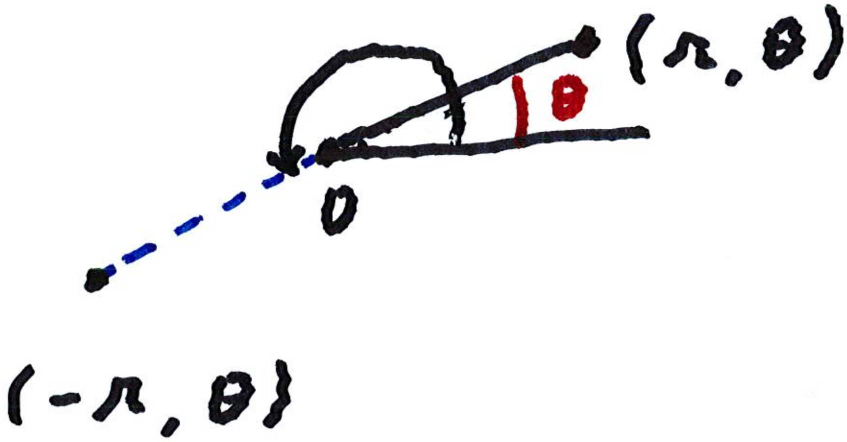
r = distance of P from the origin

and,

θ = the angle between the positive x-axis and the line OP (measured in radians)

The angle is positive if measured in the counterclockwise direction and negative if measured in the clockwise direction.

We define polar coordinates (r, θ) when r is negative. When the points $(-r, \theta)$ and (r, θ) lie on the same line through O.

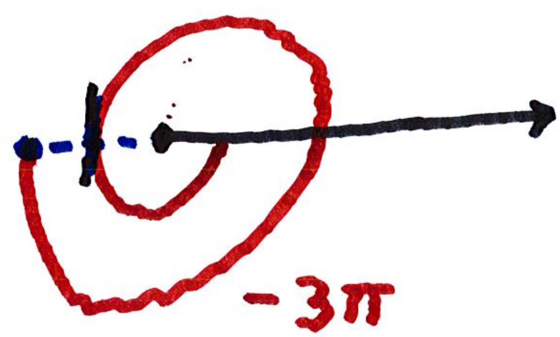


Note that $(-r, \theta)$ represents the same point as $(r, \theta+\pi)$.

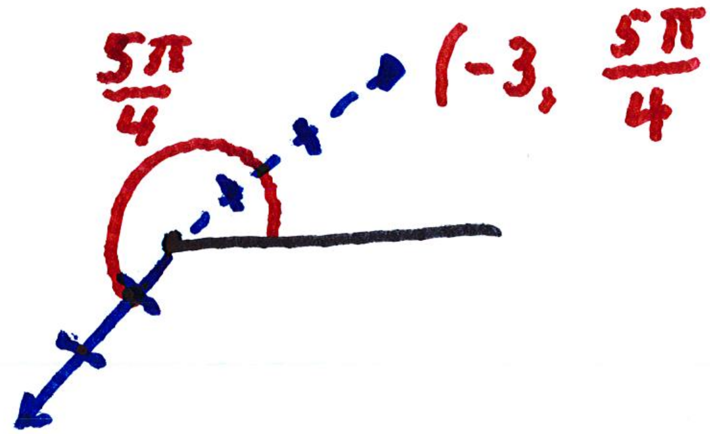
Ex. Plot the point with the given polar coordinates.

$\left(1, \frac{3\pi}{4}\right)$

$(2, -3\pi)$



$\left(-3, \frac{5\pi}{4}\right)$



In the Cartesian coordinate system, each point has only one pair of coordinates (x, y).

In polar coordinates, each point has an infite number of coordinate pairs.

For example, we can add $2n\pi$ onto θ for any integer n.

$(r, \theta + 2n\pi)$ and (r, θ)

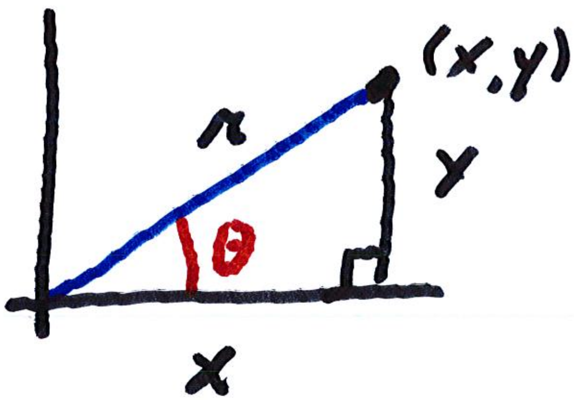
represent the same point.

Also,

$(-r, \theta + \pi)$ and (r, θ)

represent the same point,

as does $(-r, \theta + (2n+1)\pi)$



Connection between polar and Cartesian coordinates.

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

Hence

$$x = r\cos\theta \qquad y = r\sin\theta$$

These coordinates are valid for all r and all θ.

To find r and θ when

x and y are known, we use

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

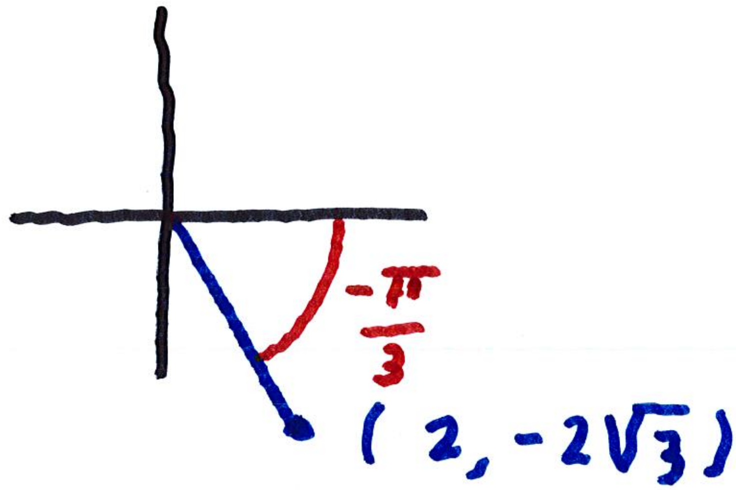
Ex. Find the polar coordinates

of the point with Cartesian

coordinates $(\underset{x}{2}, \underset{y}{-2\sqrt{3}})$

$$r^2 = 4 + 4\cdot 3 = 16$$

We choose r to be > 0.

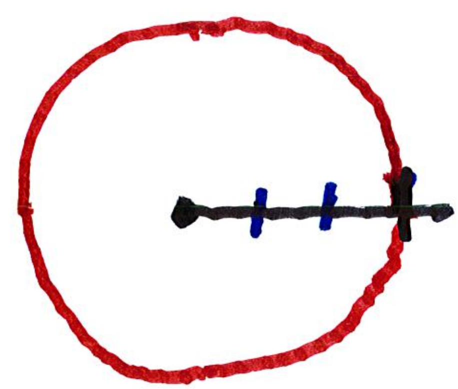


$$\tan\theta = y/x = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$$

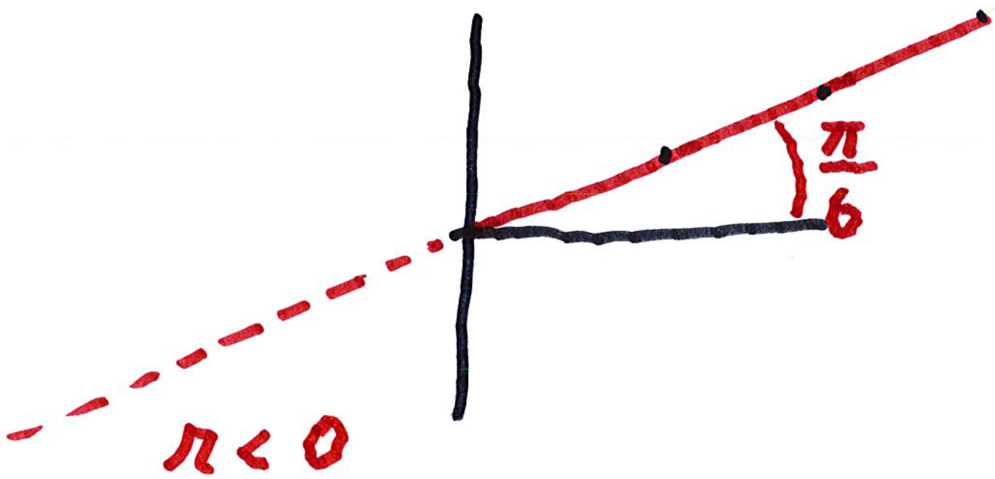
$$\therefore (r, \theta) = \left(4, \frac{-\pi}{3}\right)$$

Polar curves

Sketch $r = 3$



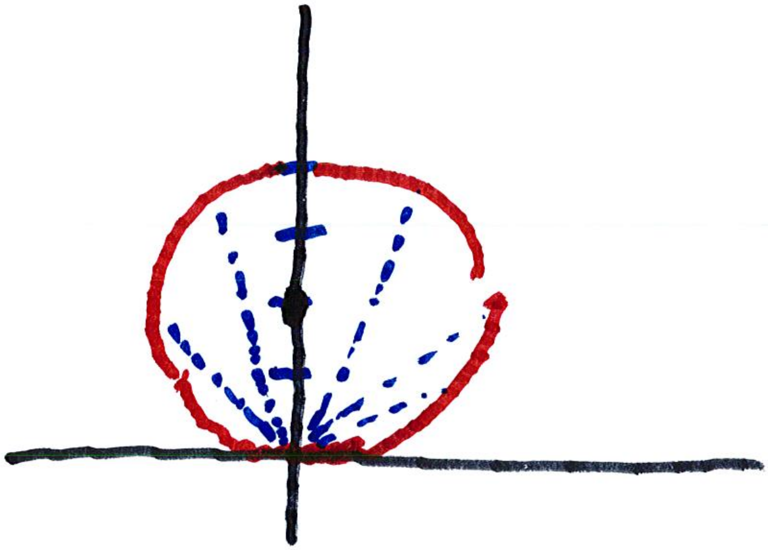
Ex. Sketch $\theta = \frac{\pi}{6}$



Ex. Sketch $r = 4\sin\theta$

θ	$r = 4\sin\theta$
0	0
$\frac{\pi}{6}$	$4 \cdot \frac{1}{2} = 2$
$\frac{\pi}{4}$	$4 \cdot \frac{1}{\sqrt{2}} = 2\sqrt{2}$
$\frac{\pi}{3}$	$4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$

$\frac{\pi}{2}$	$4 \cdot 1 = 4$
$\frac{2\pi}{3}$	$4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$
$\frac{3\pi}{4}$	$4 \cdot \frac{1}{\sqrt{2}} = 2\sqrt{2}$



Looks like a circle:

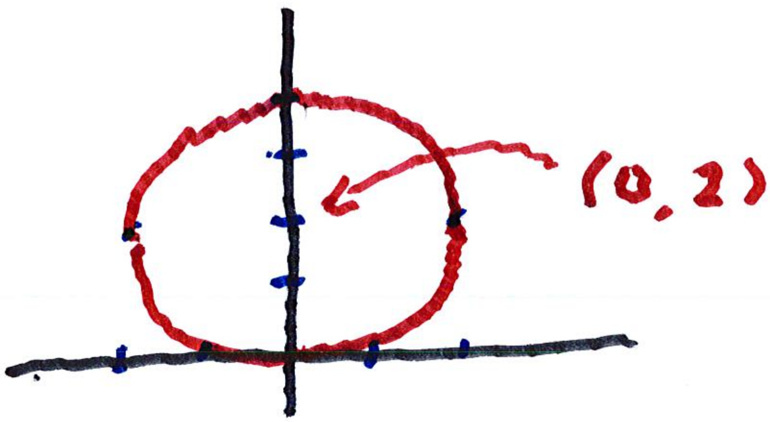
$$\sin\theta = \frac{y}{r}$$

$$\therefore \quad r = 4 \cdot \frac{y}{r}$$

$$r^2 = 4y$$

$$x^2 + y^2 = 4y \rightarrow x^2 + (y-2)^2 = 4$$

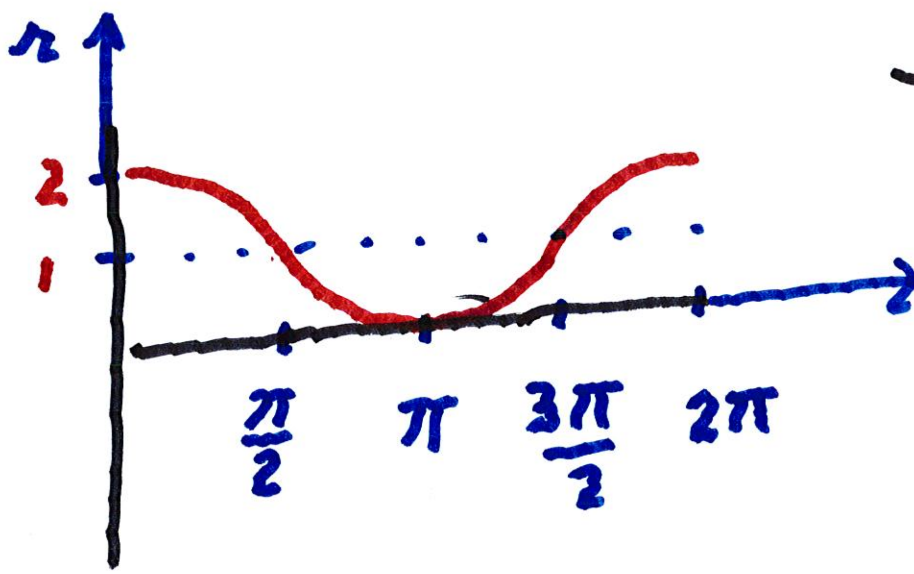
circle of radius 2 about $(0,2)$

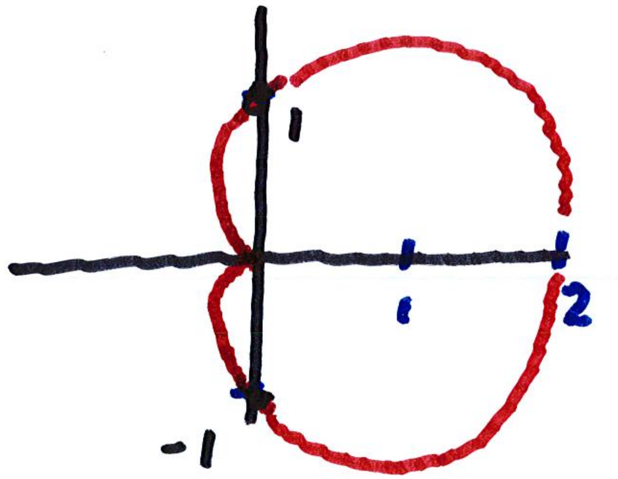


Sketch $r = 1 + \cos\theta$

First write equation in

Cartesian coord.





cardioid

(heart-shaped)

Ex. Find the polar equation for the curve $4y^2 = x$

$y = r\sin\theta \quad x = r\cos\theta$

$$4r^2\sin^2\theta = r\cos\theta$$

$$4r\sin^2\theta = \cos\theta$$

$$r = \frac{1}{4}\ \frac{\cos\theta}{\sin^2\theta}$$

$$r = \frac{1}{4}\ \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta}$$

$$\rightarrow\ r = \frac{1}{4} \cot\theta \cdot \csc\theta$$

Ex Find the polar equation

for the curve $xy = 4$

$$\rightarrow\ r\cos\theta \cdot r\sin\theta = 4$$

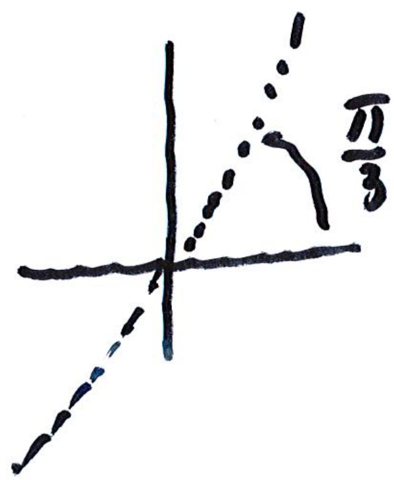
$$r^2 \cdot 2\sin\theta\cos\theta = 8$$

$$r^2 = \frac{8}{\sin 2\theta}$$

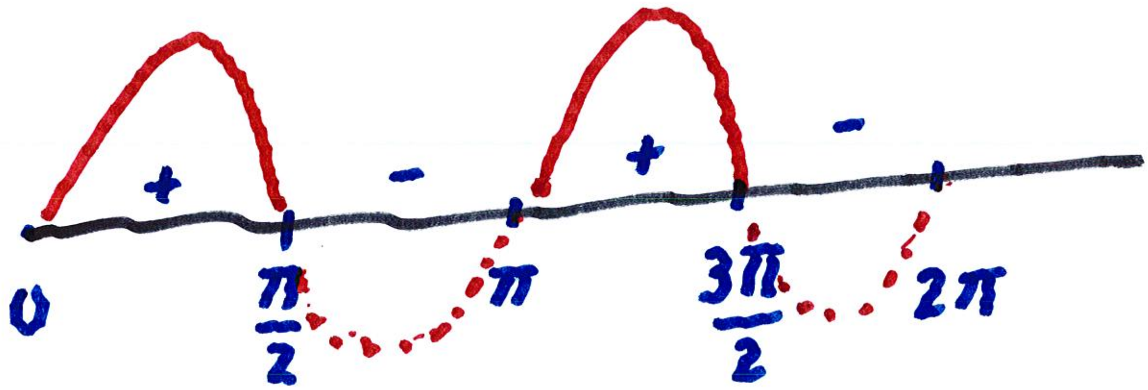
Ex Express the curve $\theta = \frac{\pi}{3}$ in Cartesian coordinates

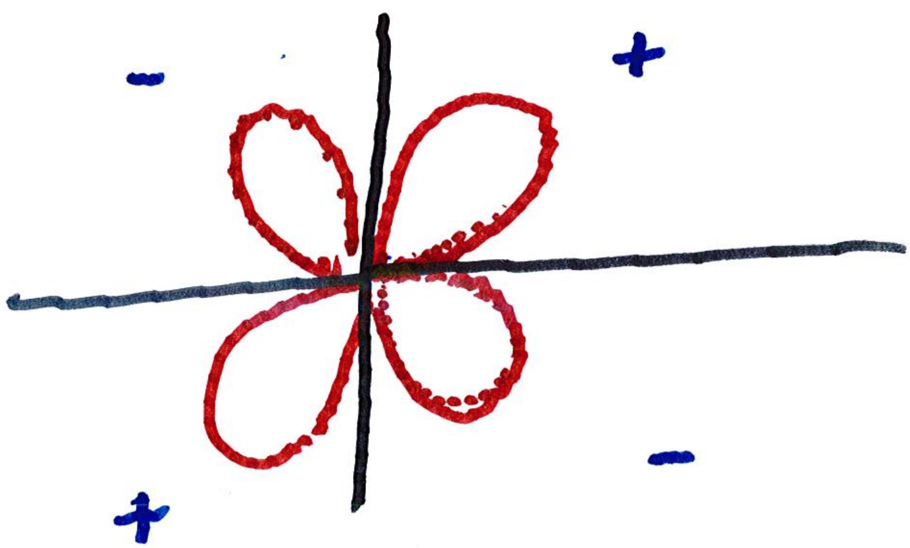
$$\frac{y}{x} = \tan\theta = \sqrt{3}$$

$$\rightarrow \underline{\underline{y = \sqrt{3}\,x}}$$



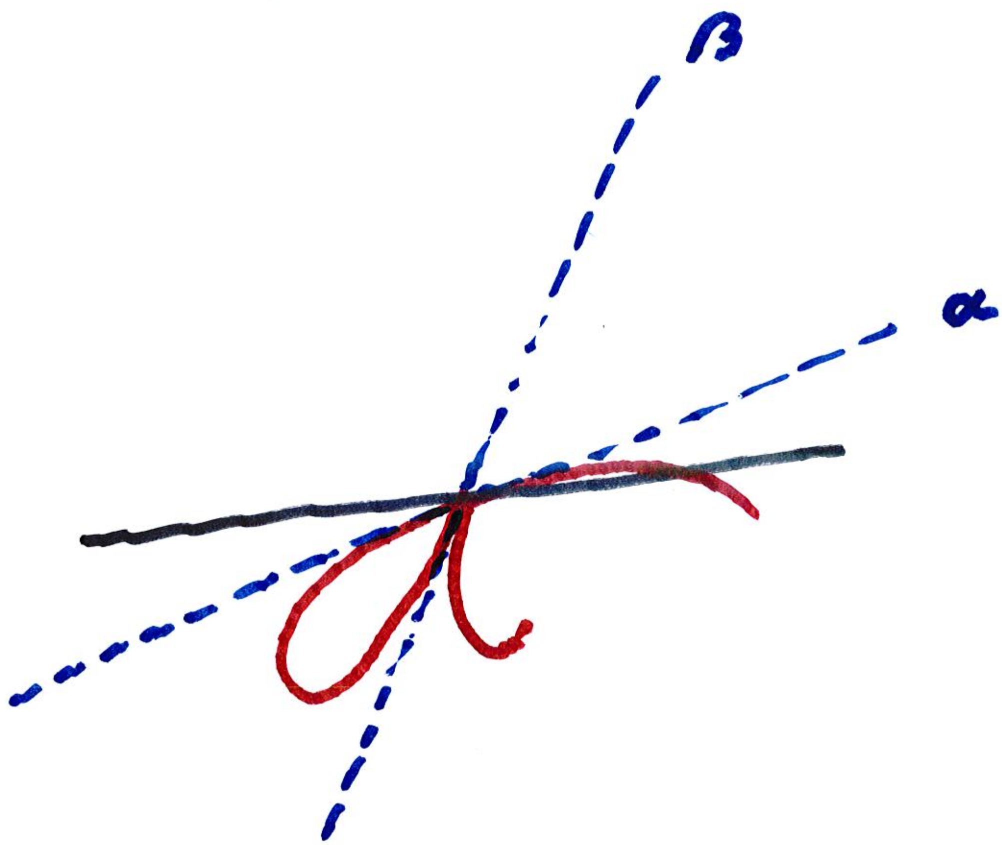
Ex Sketch the curve $r = \sin 2\theta$





Suppose $r = f(\theta)$ looks like

$f < 0$ if $\alpha < \theta < \beta$



Polar coord. graph has a loop between α and β

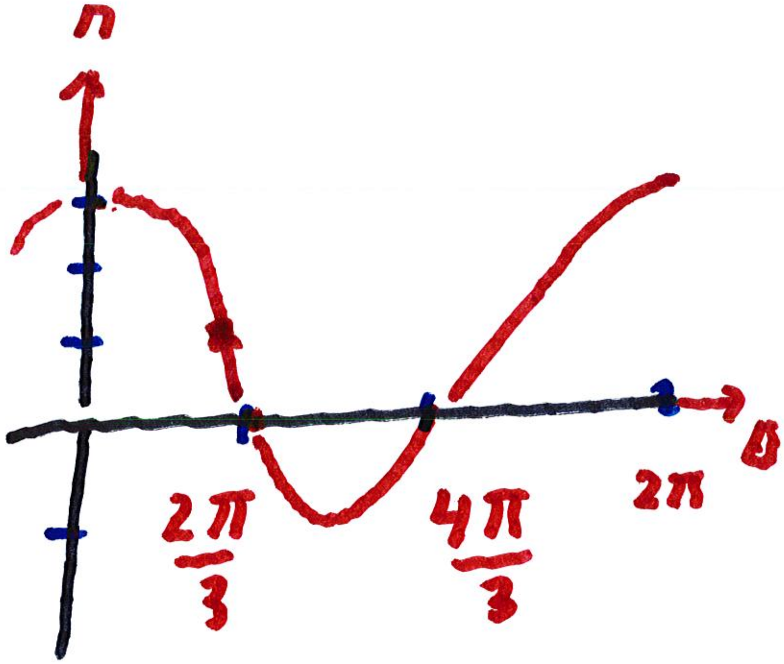
Ex. $r = 1 + 2\cos\theta$

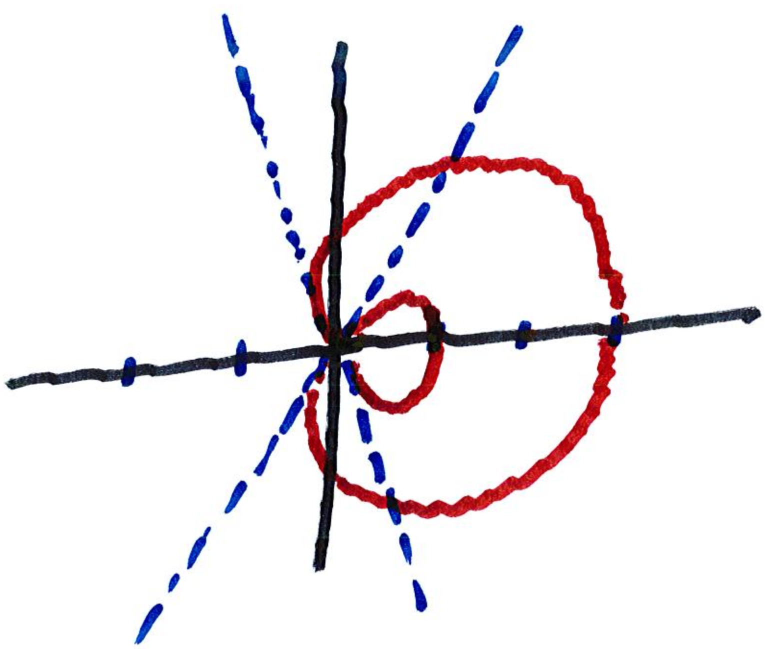
$0 = 1 + 2\cos\theta$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}$

and $\theta = \frac{4\pi}{3}$





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