10.3 Polar Coordinates

If $P$ is a point in the plane, we define the polar coordinates of $P$ by $(r, \theta)$, where

$r =$ distance of $P$ from the origin

and,

$\theta =$ the angle between the positive $x$-axis and the line $OP$ (measured in radians)

The angle is positive if measured in the counterclockwise direction
and negative if measured in the clockwise direction.

We define polar coordinates \((r, \theta)\) when \(r\) is negative. When the points \((-r, \theta)\) and \((r, \theta)\) lie on the same line through 0.
Note that \((-r, \theta)\) represents the same point as \((r, \theta + \pi)\).

Ex. Plot the point with the given polar coordinates.

\[(1, \frac{3\pi}{4})\]

\[(2, -3\pi)\]
In the Cartesian coordinate system, each point has only one pair of coordinates \((x, y)\).

In polar coordinates, each point has an infinite number of coordinate pairs.
For example, we can \(2n\pi\) onto \(\theta\) for any integer \(n\).

\((r, \theta + 2n\pi)\) and \((r, \theta)\) represent the same point.

Also,

\((-r, \theta + \pi)\) and \((r, \theta)\) represent the same point, as does \((-r, \theta + (2n+1)\pi)\)
Connection between polar and Cartesian coordinates.

\[ \cos \theta = \frac{x}{r} \quad \sin \theta = \frac{y}{r} \]

Hence

\[ x = r \cos \theta \quad y = r \sin \theta \]

These coordinates are valid for all \( r \) and all \( \theta \).
To find $r$ and $\theta$ when $x$ and $y$ are known, we use

$$r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x}$$

Ex. Find the polar coordinates of the point with Cartesian coordinates $(2, -2\sqrt{3})$

$$r^2 = 4 + 4\cdot3 = 16$$

We choose $r$ to be $> 0$. 
\[
\tan \theta = \frac{y}{x} = \frac{-2\sqrt{3}}{2} = -\sqrt{3}
\]

\[
\therefore (r, \theta) = (4, -\frac{\pi}{3})
\]

Polar curves

Sketch $r = 3$
Ex. Sketch $\theta = \frac{\pi}{6}$

Ex. Sketch $r = 4 \sin \theta$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$r = 4 \sin \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{\pi}{6}$</td>
<td>$4 \cdot \frac{1}{2} = 2$</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>$4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$</td>
</tr>
<tr>
<td>$\frac{\pi}{3}$</td>
<td>$4 \cdot \frac{1}{2} = 2\sqrt{3}$</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>$4 \cdot 1 = 4$</td>
</tr>
<tr>
<td>$\frac{2\pi}{3}$</td>
<td>$4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$</td>
</tr>
<tr>
<td>$\frac{3\pi}{4}$</td>
<td>$4 \cdot \frac{1}{2} = 2\sqrt{3}$</td>
</tr>
</tbody>
</table>
Looks like a circle:

\[ \sin \theta = \frac{y}{r} \]

\[ \therefore \quad r = 4 \cdot \frac{y}{x} \]

\[ r^2 = 4y \]

\[ x^2 + y^2 = 4y \quad \rightarrow \quad x^2 + (y-2)^2 = 4 \]

circle of radius 2 about \((0, 2)\)
Sketch $r = 1 + \cos \theta$

First write equation in Cartesian coorsd.
cardioid
(heart-shaped)

Ex. Find the polar equation for the curve $4y^2 = x$

$y = r \sin \theta$  \hspace{1cm} $x = r \cos \theta$

$4r^2 \sin^2 \theta = r \cos \theta$

$4r \sin^2 \theta = \cos \theta$
\[ r = \frac{1}{4} \frac{\cos \theta}{\sin^2 \theta} \]

\[ r = \frac{1}{4} \frac{\cos \theta}{\sin \theta} \cdot \frac{1}{\sin \theta} \]

\[ \Rightarrow r = \frac{1}{4} \cot \theta \cdot \csc \theta \]

Ex: Find the polar equation for the curve \( xy = 4 \)

\[ \Rightarrow \rho \cos \theta \cdot \rho \sin \theta = 4 \]

\[ \rho^2 \cdot 2 \sin \theta \cos \theta = 8 \]

\[ \rho^2 = \frac{8}{\sin 2 \theta} \]
Express the curve \( \theta = \frac{\pi}{3} \) in Cartesian coordinates.

\[
\frac{y}{x} = \tan \theta = \sqrt{3}
\]

\[\rightarrow y = \sqrt{3}x\]
Ex Sketch the curve \( r = \sin 2\theta \)
Suppose \( n = f(\theta) \) looks like

\[ f < 0 \text{ if } \alpha < \theta < \beta \]

Polar coord. graph has a loop between \( \alpha \) and \( \beta \)
Ex. \( r = 1 + 2 \cos \theta \)

\[
\begin{align*}
\theta &= \frac{2\pi}{3} \\
\cos \theta &= -\frac{1}{2} \\
\end{align*}
\]

and \( \theta = \frac{4\pi}{3} \)

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